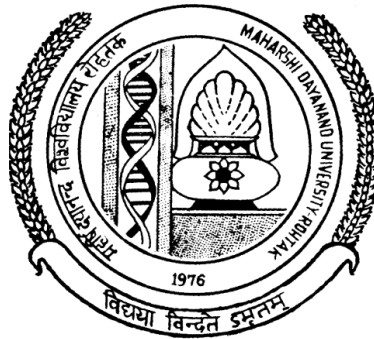


M. Sc. Mathematics (DDE)

Semester – II

Paper Code – 20MAT22C5

OPERATIONS RESEARCH TECHNIQUES



DIRECTORATE OF DISTANCE EDUCATION

MAHARSHI DAYANAND UNIVERSITY, ROHTAK

(A State University established under Haryana Act No. XXV of 1975)

NAAC 'A+' Grade Accredited University

Material Production

Content Writer: *Dr. Monika*

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MASTER OF SCIENCE (MATHAMATICS)

Second Semester

Paper code: 16MAT22C5

Operations Research Techniques

M. Marks = 100

Term End Examination = 80

Assignment = 20

Time = 3 hrs

Course Outcomes

Students would be able to:

CO1 Identify and develop operations research model describing a real-life problem.

CO2 Understand the mathematical tools that are needed to solve various optimization problems.

CO3 Solve various linear programming, transportation, assignment, queuing, inventory and game problems related to real life.

Section - I

Operations Research: Origin, Definition and scope.

Linear Programming: Formulation and solution of linear programming problems by graphical and simplex methods, Big - M and two-phase methods, Degeneracy, Duality in linear programming.

Section - II

Transportation Problems: Basic feasible solutions, Optimum solution by stepping stone and modified distribution methods, Unbalanced and degenerate problems, Transshipment problem. Assignment problems: Hungarian method, Unbalanced problem, Case of maximization, Travelling salesman and crew assignment problems.

Section - III

Concepts of stochastic processes, Poisson process, Birth-death process, Queuing models: Basic components of a queuing system, Steady-state solution of Markovian queuing models with single and multiple servers (M/M/1, M/M/C, M/M/1/k, M/MC/k)

Section - IV

Inventory control models: Economic order quantity (EOQ) model with uniform demand, EOQ when shortages are allowed, EOQ with uniform replenishment, Inventory control with price breaks.

Game Theory: Two-person zero sum game, Game with saddle points, the rule of dominance; Algebraic, Graphical and linear programming methods for solving mixed strategy games.

Note: The question paper of each course will consist of **five** Sections. Each of the sections **I to IV** will contain **two** questions and the students shall be asked to attempt **one** question from each. **Section-V** shall be **compulsory** and will contain **eight** short answer type questions without any internal choice covering the entire syllabus.

Books recommended:

1. H.A. Taha, Operation Research-An introduction, Printice Hall of India.
2. P.K. Gupta and D.S. Hira, Operations Research, S. Chand & Co.
3. S.D. Sharma, Operation Research, Kedar Nath Ram Nath Publications.
4. J.K. Sharma, Mathematical Model in Operation Research, Tata McGraw Hill.

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1

INTRODUCTION TO OPERATIONS RESEARCH

Structure

- 1.1. Introduction
- 1.2. Origin and Definitions of Operations Research
- 1.3. Scope of Operations Research
- 1.4. Advantages of Operations Research
- 1.5. Limitations of Operations Research
- 1.6. Convex Set
- 1.7. Check Your Progress
- 1.8. Summary

1.1. INTRODUCTION

Operations Research (O.R.) is a discipline that provides scientific methods for the purpose of solving real life problems that helps us in determining the best utilization of limited resources. Here we study about optimization techniques. In everyday life, we observe many situations of optimization around us. For example, suppose we want to maximize the profit or minimize the cost then maximization of the profit or minimization of cost is the optimization of profit/cost. In O.R., we obtain the optimal solution for decision making problems with the help of optimization techniques. This chapter contains origin, definitions and scope of Operations Research. In this unit, we also discuss the concept of convex sets.

1.1.1. Objectives. The objective of these contents is to get familiar reader with Operations Research. After studying this unit, reader should be able to define/describe the following concepts like:

- What is Operations Research?
- Origin of Operations Research.
- Scope of operation research.
- Convex Set.

1.2. ORIGIN AND DEFINITIONS OF OPERATIONS RESEARCH

Origin: Operations Research came into existence and gained prominence during the World War II in Britain with the establishment of team of scientists to study the strategic and tactical problems of various military operations. Scientists of different disciplines were part of this team, their research on military operation soon find applications in other fields also. Now, it was started applying in the fields of industry, trade, agriculture, planning and various other fields of economy and named as

'Operations Research'. Hence the scientific methods and techniques of Operations Research became equally useful for the planners, economists, administrators, irrigation or agricultural experts and statisticians etc. The use of Operations Research has not limited to the Britain only. Many countries of the world had started using O.R. India was one of the few first countries who started using O. R. Regional Research Laboratory located at Hyderabad was the first Operations Research unit established in India during 1949. With the opening of this unit Operations Research in India came into existence. At the same time one more unit was set up in Defence Science Laboratory. In 1955, Operations Research Society of India was formed. Today, O.R. became a professional discipline and studied as a popular subject in Management institutes and school of Mathematics.

Definitions:

Operations Research can be defined simply as combination of two words operation and research where operation means some action applied in any area of interest and research imply some organized process of getting and analysing information about the problem environment. However, many scientists or experts has been defined O.R. in various ways but the opinions about the definitions of it have been changed according to the growth of the subject. So before defining O.R. it is important to see few definitions of it.

1. O.R. is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.

-Morse and Kimbal (1946)

2. O.R. is a scientific method of providing executive with an analytical and objective basis for decisions.

-P.M.S. Blackett (1948)

3. O.R. is the application of scientific methods, techniques and tools to problems involving the operations of system so as to provide these in control of the operations with optimum solutions to the problem.

-Churchman, Acoff, Arnoff (1957)

4. O.R. is a management activity pursued in two complementary ways one-half by the free and bold exercise of commonsense untrammelled by any routine, and other half by the application of a repertoire of well-established pre created methods and techniques.

-Jagjit Singh (1968)

On the basis of all above opinions, Operations Research can be defined in more general and comprehensive way as:

“Operation research is a branch of science which is concerned with the application of scientific methods and techniques to decision making problems and with establishing the optimal solutions”.

1.3. SCOPE OF OPERATIONS RESEARCH

Scope of O.R. is very wide in today’s world as it provides better solution to various decision-making problems with great speed and efficiency. Areas where methods/models developed in Operations Research can be applied are given here under:

1. In Agriculture:

With the explosion of population and consequent shortage of food, every country is facing the problem of optimum allocation of land to various crops in accordance with the climatic conditions, optimum distribution of water from different resources. Problems of agriculture production under various restrictions can be solved by applications of Operations Research techniques.

2. In Defence Operations:

Since Second World War operation research have been used for Defence operations with the aim of obtaining maximum gains with minimum efforts.

3. In Finance:

In these modern times, government of every country or every organisation wants to introduce such type of planning/policies regarding their finance and accounting which optimize capital investment, determine optimal replacement strategies, apply cash flow analysis for long range capital investments, formulate credit policies, credit risk. Techniques developed in O.R. can be applied for attaining above said things.

4. In Marketing:

A Marketing Administrator has to face many problems like production selection, formulation of competitive strategies, distribution strategies, selection of advertising media with respect to cost and time, finding the optimal number of salesmen, finding optimum time to launch a product. All such problems can be overcome using Operations Research Techniques.

5. In Personnel Management:

Every organization wants to make selection of personnel on minimum salary. It needs to find the best combination of workers in different categories with respect to costs, skills, age and nature of jobs. It also needs to frame recruitment policies, assign jobs to machines or workers.

6. In LIC:

Operations Research Techniques can be fruitfully applied in LIC offices as it enables the policy makers to decide the premium rates for various modes of policies.

7. In Research and Development:

In determination of the areas of concentration of research and development. It also helps in project selection.

O.R. helps in solving many other problems faced by public as well as private sectors such as the ones in economic and social planning, management of natural resources, energy, housing pollution control, waiting lines and administrative problems, insurance policies and many more.

1.4. ADVANTAGES OF OPERATIONS RESEARCH

Following are certain advantages of Operations Research (OR):

- Operations Research helps decision –maker to take better and quicker decisions. It helps decision –maker to evaluate the risk and results of all the alternative decisions. So, it improves the quality of decisions and makes the decisions more effective.
- Operation Research helps, in preparing future managers as it provides in-depth knowledge about a particular action.
- Operations Research develop models, which provides logical and systematic approach for understanding, Solving and controlling a problem.
- Operations research reduces the chances of failure as it provides many alternatives for one problem, which helps the management to choose the best decision. Even managers can evaluate the risks associated with each solution and can decide whether they want to go with the solution or not.
- It helps users in optimum use of resources. For example, linear programming techniques in Operations Research suggest most effective methods and efficient ways of optimality.
- It helps in finding the limitations and scope of an activity.
- Using this information, he can measure the performance of employees and can compare it with the standard performance. It modifies mathematical solutions before these are applied. Managers may accept or modify the mathematical solutions obtained using Operations Research techniques.
- It helps suggest alternative solutions for the same optimum profit if the management wants so.

1.5. LIMITATIONS OF OPERATIONS RESEARCH

- Formulation of mathematical models may take into account all possible factors for defining a real-life problem and hence is difficult. As a result, the help of computers is required for the large number of cumbersome computations for such problems. This discourages small companies and other organisations from using O.R. techniques.
- Unquantifiable factors: Some problems may involve a large number of intangible factors such as human emotions, human relationship, etc. which cannot be quantified. Hence, the best solution cannot be determined for such problems because such factors have to be excluded.
- Dependence on experts: A specialist, who may be a mathematician or a statistician, is needed to understand the formulation of models, find solutions and recommend their implementation. Managers, who deal with such problems, may not have such specialisation. Managers, who deal with such problems, may not have such specialisation and hence the results may not be optimal.
- Model is abstraction of real-life situations and not the reality.
- Assumptions need to be made about the nature and importance of some factors in order to construct an Operation Research model.

- A reasonably good solution without the use of Operation Research may be preferred by the management as compared to a slightly better solution provided by using Operation Research since it is very expensive in terms of time and money.

In the next chapter onwards, we shall introduce various O.R. techniques for obtaining optimal and feasible solutions. Before studying these techniques, you must familiar with some important basic concepts like convex sets and basic feasible solutions. Now, we will discuss these concepts.

1.6. CONVEX SET

A region or a set K is convex if and only if for any two points on the set K , the line segment connecting these points lies entirely in K . Mathematically, $(x_1, y_1) \in K$.

$$(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \in K \quad , 0 \leq \lambda \leq 1$$

Where $(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2)$ gives all the points which lie on the line segment joining (x_1, y_1) and (x_2, y_2) .

Example of a Convex Set

1.6.1. Example.

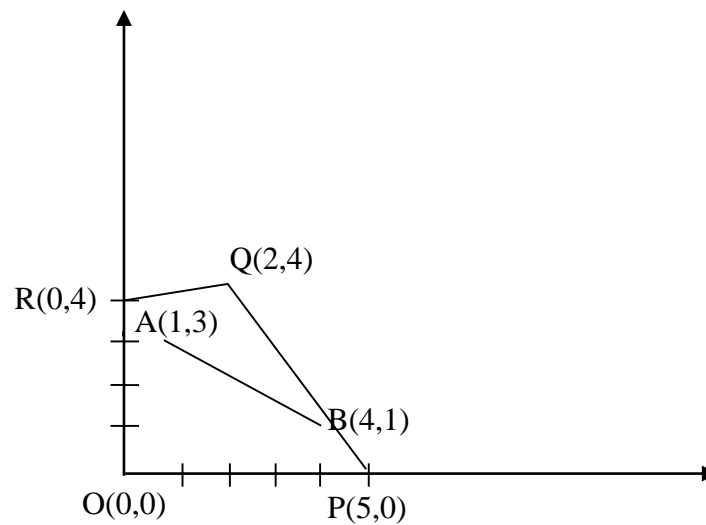


Fig. 1.1

Consider the region enclosed by $OPQR$. Let us denote it by K . It is convex as the line segment joining any two points in this region lies wholly within it. As an example, let us take two points $A(1, 3)$ and $B(4, 1)$. Then all points on the line segment joining A and B are given by

$$\begin{aligned} (\lambda(1) + (1 - \lambda)4, \lambda(3) + (1 - \lambda)(1)) &= (\lambda + 4 - 4\lambda, 3\lambda + 1 - \lambda) \\ &= (4 - 3\lambda, 1 + 2\lambda), \quad 0 \leq \lambda \leq 1. \end{aligned}$$

Here $\lambda = 0$ gives the point $Q(4, 1)$ and $\lambda = 1$ gives the point $P(1, 3)$. Other points on the line segment AB are given by

$$= (4 - 3\lambda, 1 + 2\lambda), \text{ where } 0 < \lambda < 1$$

For example, let us take $\lambda=0.1, 0.3, 0.5, 0.7, 0.9$; then the corresponding points (after substituting the values of $\lambda = 0.1, 0.3, 0.5, 0.7, 0.9$) are;

$(4 - 3(0.1), 1 + 2(0.1)), (4 - 3 \times 0.3, 1 + 2 \times 0.3), (4 - 3 \times 0.5, 1 + 2 \times 0.5), (4 - 3 \times 0.7, 1 + 2 \times 0.7), (4 - 3 \times 0.9, 1 + 2 \times 0.9)$

i.e., $(3.7, 1.2), (3.1, 1.6), (2.5, 2), (1.9, 2.4), (1.3, 2.8)$

All these points clearly lie on the line and also in the region K.

Similarly, all other points on the line segment AB also lie inside the region K. Hence, the line segment AB lies in K.

Therefore, K is convex in this example.

Example of Non-Convex Set

1.6.2. Example.

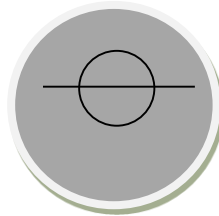


Fig. 1.2

Consider the shaded region in Fig. 1.2, clearly the line segment joining two points do not lie wholly in the region and hence this is an example of non-convex set.

1.6.3. Example. Show that the set $T = \{(x, y): x^2 + y^2 \leq 1\}$ is a convex set.

Solution: Let us take any two points A (x_1, y_1) and B (x_2, y_2) in Fig. 1.3 such that:

$x_1^2 + y_1^2 \leq 1$, And $x_2^2 + y_2^2 \leq 1$.

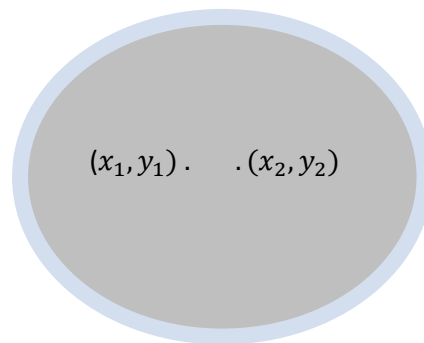


Fig. 1.3

Now, the line segment joining A and B is the set

$\{\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2: 0 \leq \lambda \leq 1\}$.

Let $u_1 = \lambda x_1 + (1 - \lambda)x_2, u_2 = \lambda y_1 + (1 - \lambda)y_2$

Therefore, all points on the line segment AB are given by (u_1, u_2) .

Now, the line segment AB lies wholly in T if

$$u_1^2 + u_2^2 \leq 1$$

$$\begin{aligned} \text{Since } u_1^2 + u_2^2 &= [\lambda x_1 + (1 - \lambda)x_2]^2 + [\lambda y_1 + (1 - \lambda)y_2]^2 \\ &= \lambda^2 x_1^2 + (1 - \lambda)^2 x_2^2 + 2\lambda(1 - \lambda)x_1 x_2 + \lambda^2 y_1^2 + (1 - \lambda)^2 y_2^2 + 2\lambda(1 - \lambda)y_1 y_2 \\ &= \lambda^2 [x_1^2 + y_1^2] + (1 - \lambda)^2 [x_2^2 + y_2^2] + 2\lambda(1 - \lambda)[x_1 x_2 + y_1 y_2] \end{aligned}$$

We have

$$u_1^2 + u_2^2 \leq \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda)[x_1 x_2 + y_1 y_2] \quad \dots(1)$$

$$\begin{aligned} \text{Now consider } (x_1 x_2 + y_1 y_2)^2 &= x_1^2 x_2^2 + y_1^2 y_2^2 + 2x_1 x_2 y_1 y_2 \\ &= x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2 - x_1^2 y_2^2 - x_2^2 y_1^2 + 2x_1 x_2 y_1 y_2 \\ &= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - x_1 y_2 (x_1 y_2 - x_2 y_1) - x_2 y_1 (x_2 y_1 - x_1 y_2) \\ &= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - (x_2 y_1 - x_1 y_2)^2 \\ &\leq (x_2 y_1 - x_1 y_2) \leq 1 \\ &\Rightarrow (x_1 x_2 + y_1 y_2) \leq 1 \end{aligned}$$

\therefore From (1) and (2), we have

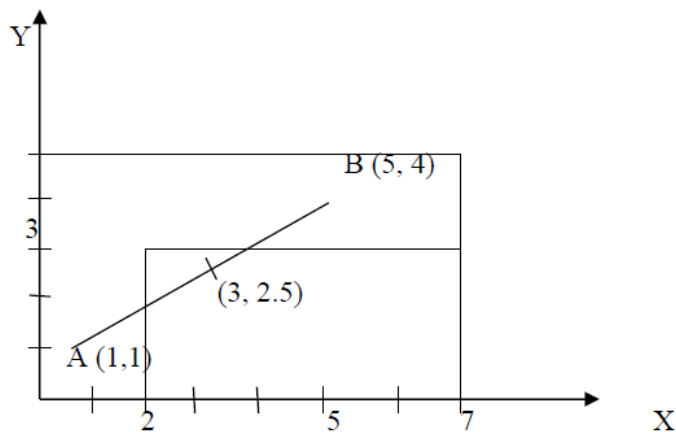
$$u_1^2 + u_2^2 \leq \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda)$$

$$\begin{aligned} \text{Or } u_1^2 + u_2^2 &\leq [\lambda + (1 - \lambda)]^2 \\ &\Rightarrow u_1^2 + u_2^2 \leq 1 \end{aligned}$$

\therefore T is convex set.

1.6.4. Example. Show that the set $T = \{(x, y): 0 \leq y \leq 5 \text{ when } 0 \leq x \leq 2 \text{ and } 3 \leq y \leq 5 \text{ when } 2 \leq x \leq 7\}$ is not a convex set.

Solution: Let us take two points A (1, 1) and B (5, 4) in the given region S.



Now, all the points on the line segment AB are given as

$$\begin{aligned} & \{ \lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2, 0 \leq \lambda \leq 1 \} \\ & = \{ \lambda(1) + (1 - \lambda)5, \lambda(1) + (1 - \lambda)4, 0 \leq \lambda \leq 1 \} \\ & = \{ 5 - 4\lambda, 4 - 3\lambda, 0 \leq \lambda \leq 1 \} \end{aligned}$$

Let us take one of these points and put $\lambda = \frac{1}{2}$. So, the point is

$$(5 - 4 \times \frac{1}{2}, 4 - 3 \times \frac{1}{2}) = (5 - 2, 4 - \frac{3}{2}) = (3, 2.5)$$

This point is on the line, but does not belong to the given set T since for $y = 2.5$, x should lie between 0 and 2 but here it is 3. Therefore, the given set is not convex.

1.6.5. Extreme Points of a convex set

A point (x, y) in a convex set K is called an extreme point if it is not possible to locate two distinct points in or on K so that the line joining them will include (x, y) .

Mathematically, a point (x, y) is an extreme point of a convex set if it cannot be expressed as a convex combination of any two points (x_1, y_1) and (x_2, y_2) [for $(x_1, y_1) \neq (x_2, y_2)$] in the set such that

$$x = \lambda x_1 + (1 - \lambda)x_2 \quad \text{and} \quad y = \lambda y_1 + (1 - \lambda)y_2, \quad 0 < \lambda < 1$$

Remark:

- i) The vertices of the polygons, which are convex sets, are the extreme points.
- ii) Every point on the circumference of the region containing the portion in and on the circle is an extreme point.

1.6.6. IMPORTANT DEFINITIONS

Solution

A set of values of the decision variables which satisfy the constraints of the given LPP is said to be a solution of that LPP.

Feasible Solution

A solution in which values of decision variables satisfy all the constraints and non-negativity conditions of an LPP simultaneously is known as feasible solution.

Infeasible Solution

A solution in which values of decision variables do not satisfy all the constraints and non-negativity conditions of an LPP simultaneously is known as infeasible solution.

Basic solution

Suppose there are m equations representing constraints (limited available resources) containing $m + n$ variables in an allocation problem. The solution obtained by setting any n variables equal to zero and solving for the remaining m variables and the remaining n variables are non – basic variables.

The maximum number of possible basic solutions is given by the formula C_n^{m+n}

For example, if there are 2 equations in 3 variables, then the maximum number of possible basic solutions is

$$C_2^3 = \frac{3!}{2!(3-2)!} = 3.$$

Basic Feasible Solution

A basic solution for which all the basic variables are non – negative is called the basic feasible solution.

Further basic feasible solution are of two types:

Degenerate Solution

A basic feasible solution is known as degenerate if value of at least one basic variable is zero.

Non-Degenerate Solution

A basic feasible solution is known as non- degenerate if value of all basic variables are non-zero and positive.

Optimum Basic Feasible Solution

A basic feasible solution which optimizes i.e. maximise or minimise the objective function value of the given LPP is called optimum basic feasible solution.

Unbounded Solution

A solution in which value of the objective function of the given LPP increase/decrease indefinitely is called an unbounded solution.

1.6.7. Example. Determine all the basic feasible solutions of the equations:

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$

$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

Solution: Here the maximum number of possible basic solutions is

$$C_2^4 = \frac{4!}{2!2!} = \frac{4 \times 3 \times 2}{2 \times 2} = 6$$

We obtain as follows:

Setting $x_1 = 0, x_2 = 0$, we have $2x_3 + x_4 = 3$ and $4x_3 + 6x_4 = 2$

$$\Rightarrow x_3 = 2, x_4 = -1$$

Setting $x_1 = 0, x_3 = 0$, we have $6x_2 + x_4 = 3$ and $4x_2 + 6x_4 = 2$

$$\Rightarrow x_2 = \frac{1}{2}, x_4 = 0$$

Setting $x_1 = 0, x_4 = 0$, we have $6x_3 + 2x_3 = 3$ and $4x_2 + 4x_3 = 2$

$$\Rightarrow x_2 = \frac{1}{2}, x_3 = 0$$

Setting $x_2 = 0, x_3 = 0$, we have $2x_1 + x_4 = 3$ and $6x_1 + 6x_4 = 2$

$$\Rightarrow x_1 = \frac{8}{3}, x_4 = -\frac{7}{3}$$

Setting $x_2 = 0, x_4 = 0$, we have $2x_1 + 2x_3 = 3$ and $6x_1 + 4x_4 = 2$

$$\Rightarrow x_1 = -2, x_4 = \frac{7}{2}$$

Setting $x_3 = 0, x_4 = 0$, we have $2x_1 + 6x_2 = 3$ and $6x_1 + 4x_2 = 2$

$$\Rightarrow x_1 = 0, x_2 = \frac{1}{2}$$

Thus, all basic solutions of the given system of equations are;

$$(0, 0, 2, -1), (0, \frac{1}{2}, 0, 0), (0, \frac{1}{2}, 0, 0), (\frac{8}{3}, 0, 0, -\frac{7}{3}), (-2, 0, \frac{7}{2}, 0), (0, \frac{1}{2}, 0, 0).$$

Here $(0, \frac{1}{2}, 0, 0)$ is repeated thrice and hence the basic solutions are

$$(0, 0, 2, -1), (0, \frac{1}{2}, 0, 0), (\frac{8}{3}, 0, 0, -\frac{7}{3}), (-2, 0, \frac{7}{2}, 0).$$

Of these solutions, the basic feasible solution is $(0, \frac{1}{2}, 0, 0)$ as in the other solutions, all decision variables are not positive.

1.6.8. Exercises.

1 Find all basic solutions for the system of simultaneous equations:

$$2x_1 + 3x_2 + 4x_3 = 5 \quad \text{and} \quad 3x_1 + 4x_2 + 5x_3 = 6.$$

(Hint. The maximum number of possible basic solutions is

$$C_2^3 = \frac{3!}{2!(3-2)!} = 3.$$

And no feasible solution)

2 Determine all the basic feasible solutions of the system of equations:

$$3x_1 + 5x_2 + x_3 = 15 \text{ and } 5x_1 + 2x_2 + x_4 = 10.$$

Further, discuss that whether the solution is degenerate or non-degenerate.

(Hint. The maximum number of possible basic solutions will be $C_2^4 = \frac{4!}{2!2!} = \frac{4 \times 3 \times 2}{2 \times 2} = 6$.

And the possible solutions are

$$(0, 0, 15, 10), (0, 3, 0, 4), (0, 5, -10, 0), (5, 0, 0, -15), (2, 0, 9, 0) \text{ and } (\frac{50}{19}, \frac{45}{19}, 0, 0).$$

1.7. CHECK YOUR PROGRESS

- 1 Explain the concept and scope of Operations Research.
- 2 Explain the advantages and disadvantages of Operations Research.
- 3 Define the followings:
 - (i) Feasible solution
 - (ii) Degenerate solution
 - (iii) Extreme points of a convex set.

1.8. SUMMARY

In this chapter, we have introduced Operations Research, its scope, advantages and limitations. We have observed that Operations Research is a very powerful method of getting the best out of limited resources. It finds applications in almost every field. Here, we explain concept of convex sets which is another important concept. We study feasible solution, basic solution, and basic feasible solution of a system of equations less in number than the number of decision variables. Such solutions are required to be obtained for finding out optimal solution of the given LPP.

2

LINEAR PROGRAMMING PROBLEMS

Structure

- 2.1. Introduction
- 2.2. Linear Programming Problem (LPP)
- 2.3. Mathematical Formulation of LPP
- 2.4. Graphical Method
- 2.5. Canonical and Standard Form of LPP
- 2.6. Check Your Progress.
- 2.7. Summary

2.1. INTRODUCTION

In 1947, George Dantzig and his associates, while working with the US Air Force during World War II, developed this technique, primarily for solving military logistics problems. They observed that a large number of military programming/planning problems could be formulated as maximizing/minimizing a linear form of profit/cost function whose variable were restricted to values satisfying a system of linear constraints. In chapter 1, We have already discussed the concept of optimization and explained the basic feasible solution of linear programming problem.

In this chapter, we study linear programming problems (LPP), their mathematical formulation, objective function concept and graphical method. We use graphical method mainly for solving problems involving two variables. Linear programming can be applied to a variety of problems such as production, transportation, advertising and problems in public and private organizations, e.g., business, industry,

hospitals, libraries as also in education. In order to solve linear programming problems, we need to convert them into a canonical or standard form.

2.1.1. Objectives. The objective of these contents is to provide some important concepts/results to the reader like:

- Linear programming problems, it's applications and limitations.
- Mathematical formulation of linear programming problem.
- Graphical Method
- Canonical and Standard form of an LPP

2.2. LINEAR PROGRAMMING PROBLEM (LPP)

We have already familiar with the concept of optimization. A mathematical programming is an optimization technique by which the maximum or minimum value of a function is determined under certain conditions. Mathematical programming in which constraints are expressed as linear equalities / inequalities is called linear programming.

We first introduce three basic components essential for the development of LP theory.

Decision Variables: The variables in terms of which the problem is defined.

Objective Function: A function which is to be maximised or minimised subject to the given constraints/limitations.

Constraints: There are always certain limitations on the use of resources that limit the degree to which an objective can be achieved. These limitations are known as constraints or restrictions. Constraints must be represented as linear equalities or inequalities in terms of decision variables.

Every organisation, big or small wants to find the best allocation of resources in order to optimize the objective function. We can use linear programming only if the following conditions are satisfied:

- i. Objective function should be well defined
- ii. Objective function can be expressed as a linear function of the decision variables.
- iii. There should be finite number of constraints and can be expressed as linear equalities or inequalities in terms of variables.
- iv. Decision variables should be non- negative.

2.2.1. Advantages and Limitations of an LPP

Linear programming methods are used in many fields including business and industry by almost all their departments such as production, marketing, finance etc. it's some advantages are

- Helps in attaining the optimum use of resources i.e. maximise profit and minimise costs
- Improve the quality of decisions

There are many more advantages. In spite of having many advantages and wide areas of applications, there are some limitations as well. Following are certain limitations of linear programming:

- We can apply linear programming method only if relationships are linear.

- While solving LPP, it is possible that we will get non-integral values even for those decision variables which have only integral values.
- Constraints in the linear programming methods are written assuming all parameters are known and should be constant. However, in real problems, sometimes these are neither known nor constant.
- LP deals with the problems having single objective, whereas in real-life, there are many situations where we have to achieve multi-objectives.

2.3. MATHEMATICAL FORMULATION OF LPP

In our daily life, there are many real-life situations where LP problems may arise and for using LPP methods/techniques to find a solution of such situations, it becomes necessary to present the given word problem into mathematical form correctly. The steps of mathematical formulation of LPP are summarized as follows:

- Identify the decision variable of the given problem.
- Formulate the objective function, which is to be maximised or minimised, as a linear function of the decision variables.
- Formulate the constraints and express them as linear inequalities or equalities in terms of decision variables.
- Introduce non-negative restrictions as negative values of the decision variables do not have any valid physical interpretation.

The steps of mathematical formulation of LPP are explained with the help of an example.

2.3.1. Example. A small-scale industry manufactures two products P and Q which are processed in a machine shop and assembly shop. Product P requires 2 hours of work in a machine shop and 4 hours of work in the assembly shop to manufacture while product Q requires 3 hours of work in the machine shop and 2 hours of work in the assembly shop. In one day, the industry cannot use more than 16 hours of machine shop and 22 hours of assembly shop. It earns a profit of rupees 3 per unit of product P and rupees. 4 per unit of product Q. Give a mathematical formulation of the problem as to maximise profit.

Solution: let x and y be the number of units of product P and Q, which are to be produced. Here x and y are the decision variables. Suppose Z is the profit function.

Since one unit of product P and one unit of product Q gives the profit of rupees 3 and rupees 4, respectively, the objective function is

$$\text{Maximize } Z = 3x + 4y$$

The requirement and availability in hours of each of the shops for manufacturing the products are tabulated as follows:

	Machine Shop	Assembly Shop	Profit
Product P	2 hours	4 hours	Rs.3 per unit
Product Q	3 hours	2 hours	Rs.4 per unit
Available hours per day	16 hours	22 hours	

Total hours of machine shop required for both types of product = $2x+3y$

Total hours of assembly shop required for both types of product = $4x+2y$

Hence, the constraints as per the limited available resources are:

$$2x+3y \leq 16 \text{ and } 4x+2y \leq 22$$

Since the number of units produced for both P and Q cannot be negative, the non-negative restrictions are:

$$x \geq 0, y \geq 0$$

Thus, the mathematical formulation of the given problem is:

Maximise $Z = 3x + 4y$

Subject to the constraints

$$2x + 3y \leq 16$$

$$4x + 2y \leq 22$$

And non-negative restrictions

$$x \geq 0, y \geq 0$$

2.3.2. Exercise. A company produces two types of items P and Q that require gold and silver. Each unit of type P requires 4g silver and 1g gold while that of type Q requires 1g silver and 3g gold. The company produces 8g silver and 9g gold. If each unit of type P brings a profit of rupees 44 and that of type Q rupees 55, determine the number of units of each type that the company should produce to maximise the profit.

Answer. Let Z be the profit function. The mathematical formulation of the given problem is

$$\text{Max. } Z = 44x + 55y$$

Subject to the constraints:

$$4x + y \leq 8,$$

$$x + 3y \leq 9, \quad x \geq 0, y \geq 0.$$

2.4. GRAPHICAL METHOD

The graphical method is used to solve simple linear programming problems having two decision variables. For solving LPPs involving more than two decision variables, we use another method called simplex method. We discuss it in chapter 3.

The steps of graphical method for solving an LPP are as follows:

1. Plot the graph corresponding to the given constraints.
2. Determine the region for each given constraint.
3. Determine the feasible region.
4. Determine corner/extreme points.
5. Examine corner/extreme points.

The following example explains steps of Graphical method.

2.4.1. Example.

The graphs are plotted for the equalities corresponding to the given inequalities for constraints as well as restrictions. That is, we first draw the straight lines. For example, suppose one of the given inequalities is $2x+3y \leq 6$. Then, we first plot the graph for the equation $2x+3y = 6$, which is straight line. For this, we take any two points on it join them.

For example, for $x = 0$, $y = 6/3 = 2$ and for $y = 0$, $x = 6/2 = 3$. So, we get the straight line shown in fig. 2.1.

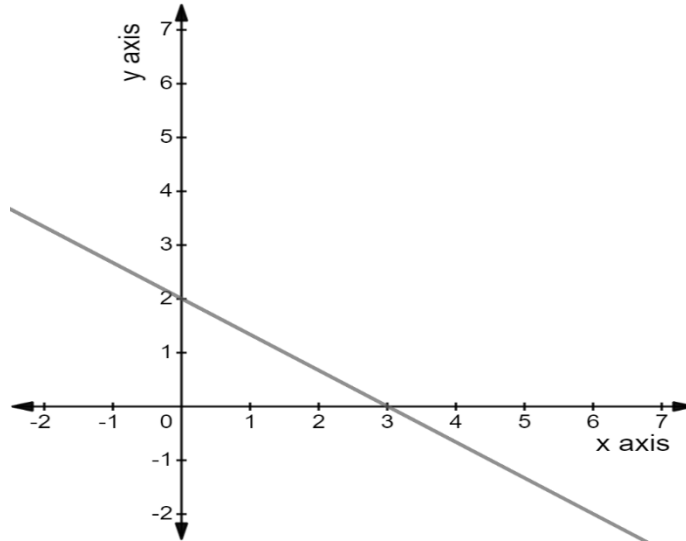


Fig.2.1

First, we determine the region corresponding to each inequality. Let us consider the inequality $2x+3y \leq 6$ again. We can find the region on the graph satisfied by this inequality by substituting $x = 0$ and $y = 0$ in it. We get

$$2(0) + 3(0) \leq 6 \Rightarrow 0 \leq 6$$

which is correct. So, it is the region containing the point $(0, 0)$. Hence, half plane shown in Fig. 2.2 by shading the region starting from the line towards the point $(0, 0)$ is the graph of the given inequality.

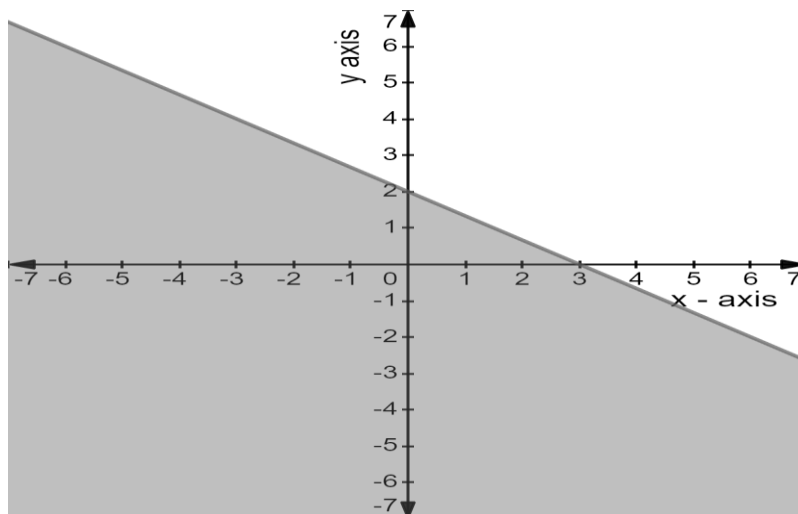


Fig.2.2

Had the given inequality been $2x+3y \geq 6$, then we would have shaded the region on the opposite side of the line. This is because on putting $x=0, y=0$ in the inequality, we get

$$2(0) + 3(0) \geq 6 \Rightarrow 0 \geq 6$$

which is not correct. Thus, the point $(0, 0)$ does not satisfy the inequality and hence does not lie in the region. Thus, graph for the inequality $2x-3y \geq 6$ would, therefore, be as shown in Fig. 2.3.

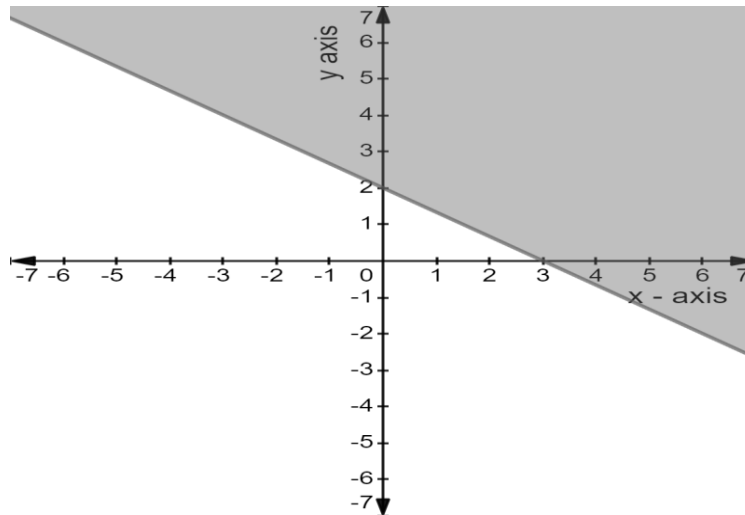


Fig.2.3

In this example, we have used the point $(0, 0)$ to determine which half plane corresponds to the given inequality. However, you can take any other point. But using $(0, 0)$ is far easier. If the right-hand side of the given inequality is zero, using the point $(0, 0)$ in it is meaningless. For example, suppose the given inequality is $2x-3y \geq 0$. The plot of $2x-3y=0$ is given in fig. 2.4. It is straight line passing through the origin. Using the point $(0, 0)$ in the inequality $2x-3y \geq 0$, we get $2(0)-3(0) \geq 0$, i.e., $0 \geq 0$. So, we cannot decide which half plane is the region of the given inequality. Therefore, in this case, we use any other point, say $(2, 0)$, on putting $x=2, y=0$ in the given inequality, we get

$$2(2)-3(0) \geq 0 \text{ and } 4 \geq 0$$

which is true. Therefore, the half plane containing $(2, 0)$ is the required region as shown in Fig.2.4.

$$2x-3y=0 \text{ or } 3y=2x \text{ or } y=2x/3$$

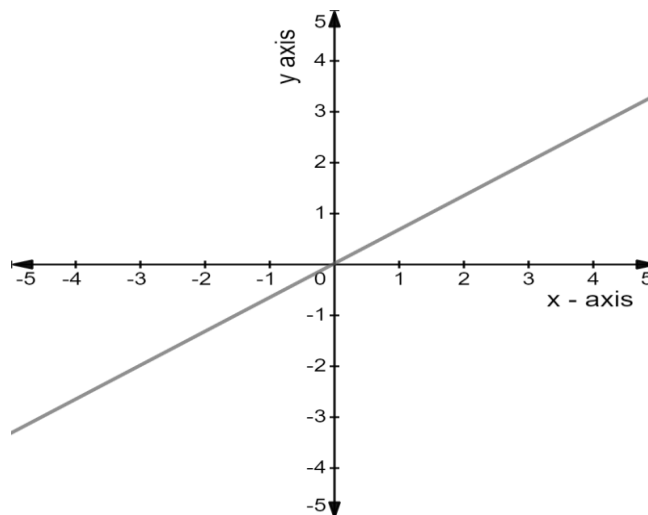


Fig.2.4

After determining the regions for each inequality, we find their common region. This is the region where all the given inequalities and non-negative restrictions are satisfied. This common region is known as feasible region or the solution set or polygonal convex set.

Now, we determine each of the corner points (vertices) of the polygon obtained in step 3. This is done either by plotting graphs on graph paper or by solving the two equations of the lines intersecting at that point.

Evaluate the value of the objective function at each corner point and determine the extreme point of the feasible region that has optimum objective function value.

According to the feasible region, we have different types of solution.

- a) If the feasible region is bounded. The maximum of the values obtained for the objective function at the corner points is the optimum value when the objective function is of maximization form. The point corresponding to this maximum value give the required values of the decision variables. The minimum of the values obtained for the objective function at the corner points is the optimum value when the objective function is of minimization form. The point corresponding to this minimum value give the required values of the decision variables.
- b) If the feasible region is not bounded. Then either there are additional hidden conditions which can be used to bound the region or there is no solution to the problem.
- c) If the same optimum value exists at two of the vertices, then there are multiple solutions to the problem. Suppose these are two points (x_1, y_1) and (x_2, y_2) . Then other solutions are given by the point as follow:

[First ordinate of first point $\times t$ + First ordinate of second point $\times (1 - t)$,

Second ordinate of first point $\times t$ + Second ordinate of second point $\times (1 - t)$]

where t is any real number lying between 0 and 1.

For example, let the objective function be $Z = 3x - y$ and let A (2,1) and B (3,4) be the points which give the same optimum value of the objective function, i.e., $Z = 5$. Then other solutions which give the same value of the objective function are:

$$(2 \times t + 3 \times (1 - t), 1 \times t + 4 \times (1 - t))$$

$$\text{Or } (2t + 3 - 3t, t + 4 - 4t)$$

$$\text{Or } (3 - t, 4 - 3t), 0 \leq t \leq 1$$

Here $t = 0$ gives the point (3, 4), which is the point B and $t = 1$ gives the point (2, 1), which is the point A. The real values of t between 0 and 1 give other points which give the same optimum solution. One such point other than A and B is

$$(3 - \frac{1}{2}, 4 - 3 \times \frac{1}{2}), t = \frac{1}{2}, \text{ i.e., } (\frac{5}{2}, \frac{5}{2})$$

You can verify that $Z = 5$ at the point $(\frac{5}{2}, \frac{5}{2})$.

2.4.2. Example. A company produces two types of items P and Q that require gold and silver. Each unit pf type P requires 4g silver and 1g gold while that od type Q requires 1g silver and 3g gold. The company produces 8g silver and 9g gold. If each unit of type P brings a profit of rupees 44 and that

of type Q rupees 55, determine the number of units pf each type that the company should produce to maximise the profit. What is the maximum profit?

Solution. Let x be the number of units of type P to be produced and y be the number of units of type Q to be produced. It is given that:

	Silver	Gold	Profit
Type P	4g	1g	Rs.44 per unit
Type Q	1g	3g	Rs.55 per unit
Available (at the most)	8g	9g	

Let Z be the profit function. The mathematical formulation of the given problem is

$$\text{Max. } Z = 44x + 55y$$

Subject to the constraints:

$$4x + y \leq 8,$$

$$x + 3y \leq 9,$$

$$x \geq 0, y \geq 0.$$

First of all, we plot the graphs for the equations:

$$4x + y = 8, x + 3y = 9, x = 0, y = 0 .$$

Since these equations are of straight lines, only two points are sufficient to plot the graphs (see fig.2.5).

For the line $4x + y = 8$, we take the following two points:

X	0	2
Y	8	0

Similarly, for the line $x + 3y = 9$, we take

X	0	9
Y	3	0

Now, plotting the above lines, we get fig.2.5.

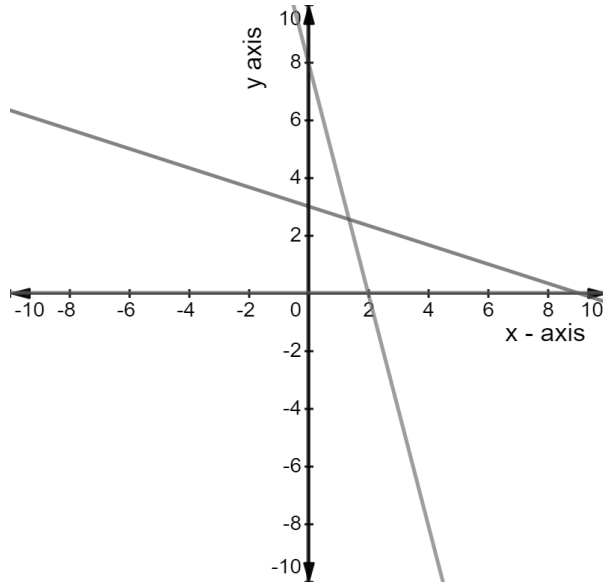


Fig.2.5

Note that $x = 0$ is the y axis and $y = 0$ is the x axis.

For plotting the graph of the inequality $4x + y \leq 8$ we put $(0, 0)$ in it. We get $0 \leq 8$, which is true. Therefore, starting from the line $4x + y = 8$, we shall shade towards origin. Similarly, for the graph $x + 3y \leq 9$, we shall shade towards origin. For the graph $x \geq 0$, we shall shade towards right side of $x = 0$ and for the graph $y \geq 0$, the region above $y = 0$ will be shaded.

Thus, the regions for the inequalities are shown in fig.2.6.

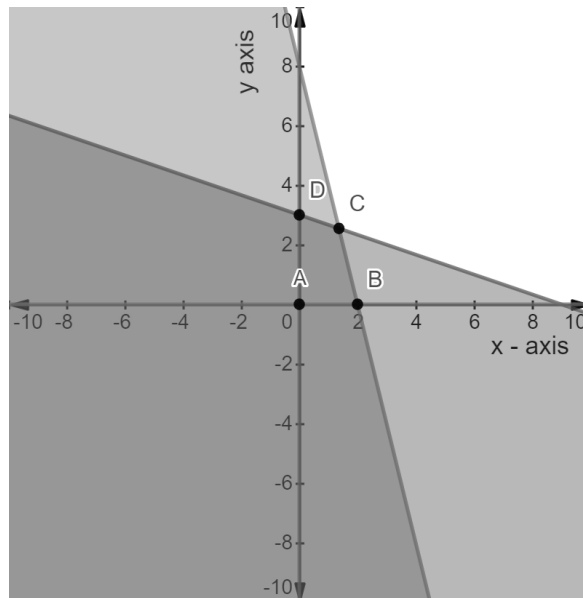


Fig.2.6.

You can see from the figure 2.6 that the coordinates of A, B, and D are $(0, 0)$, $(2, 0)$ and $(0,3)$ respectively. The coordinates of the point C are obtained by solving the equations $4x + y = 8$ and $x + 3y = 9$ as it is the point of intersection of the two lines represented by them. the solution of equations $4x + y = 8$ and $x + 3y = 9$ is given as

$$x = \frac{15}{11} \text{ and } y = \frac{28}{11}$$

So the vertices of ABCD are A(0,0) , B(2,0) , C($\frac{15}{11}$, $\frac{28}{11}$) and D (0,3).

We now obtain the values of $Z = 44x + 55y$ at each of the vertices of ABCD as follows:

At A (0,0) , $Z = 44(0) + 55(0) = 0$

At B(2,0) , $Z = 44(2) + 55(0) = 88$

At C($\frac{15}{11}$, $\frac{28}{11}$) , $Z = 44(\frac{15}{11}) + 55(\frac{28}{11}) = 60 + 140 = 200$

At D (0,3), $Z = 44(0) + 55(3) = 165$

Thus, the value of Z is maximum at C ($\frac{15}{11}$, $\frac{28}{11}$) and the optimum solution is $\text{Max.}Z = 200$ when $x = \frac{15}{11}$ and $y = \frac{28}{11}$.

2.4.3. Exercises.

1. Maximise $Z = 6X + 3Y$ subjects to the constraints $2X + 5Y \leq 120, 4X + 2Y \leq 80, X \geq 0, Y \geq 0.$

2. Maximise $z = 3x_1 + 2x_2$ subjects to the constraints $x_1 - x_2 \leq 1, x_1 + x_2 \geq 3, x_1 \geq 0, x_2 \geq 0$

3. Maximise $Z = x_1 + x_2$ subjects to the constraints $x_1 + x_2 \leq 1, -3x_1 + x_2 \geq 3, x_1 \geq 0, x_2 \geq 0$

4. A company manufactures two products X and Y, each of which requires three types of processing. The length of time for processing each unit and the profit per unit are given in the following table:

	Product X (hr/unit)	Product Y (hr/unit)	Available capacity per day (hr)
Process 1	12	12	840
Process 2	3	6	300
Process 3	8	4	480
Profit per unit	5	7	

How many units of each product should the company manufacture per day in order to maximise profit?

5. A company produces soft drinks and has a contract requiring that a minimum of 80 units of chemical A and 60 units of chemical B go into each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. The supplier X₁ has a mix of 4 units of A and 2 units of B that costs rupees 10, and the supplier X₂ has a mix of 1 unit of A and 1 unit of B that costs rupees 4. How many mixes from the company X₁ and company X₂ should the company purchase to honour contract requirement and yet minimise cost?

2.5. CANONICAL AND STANDARD FORM OF AN LPP

After formulating a linear programming problem, our next step is to solve it. You have learnt that linear programming problems can be represented as problems of maximisation or minimisation with constraints such as $\leq, =, \geq$. In order to develop a standard procedure for solving LPPs, we need to convert them into well - known form. We now discuss the General LPP along with these two forms. The canonical form is especially used in the duality theory and the standard form is used to develop the general procedure for solving any linear programming problem. In order to understand these forms, you also need to learn about slack and surplus variable.

2.5.1. General Linear Programming Problem

Let us formulate the general linear programming problem. Let Z be a linear function of n basic variables $X_1, X_2, X_3, \dots, X_n$, which is to be maximised (or minimised). We write the problem as

$$\text{Maximise (or minimise) } Z = C_1X_1 + C_2X_2 + C_3X_3 + \dots + C_nX_n \dots \text{ (1)}$$

where $C_1, C_2, C_3, \dots, C_n$ are known constant termed as **cost coefficients** of basic variables.

Let (a_{ij}) be an $m \times n$ real matrix of $m \times n$ constants a_{ij} 's and let $\{b_1, b_2, \dots, b_m\}$ be a set of constants such that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq \text{ or } = \text{ or } \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq \text{ or } = \text{ or } \geq b_2 \\ \cdot & \quad \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \quad \quad \quad \dots \text{ (2)} \\ \cdot & \quad \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\ \cdot & \quad \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq \text{ or } = \text{ or } \geq b_m \\ \text{and} \quad \quad \quad x_j &\geq 0 \text{ for all } j = 1, 2, 3, \dots, n \quad \dots \text{ (3)} \end{aligned}$$

The linear function Z in equation (1) is called the objective function. The set of inequalities given in (2) is called constraints of a general LPP and the set of inequalities given in (3) are known as non – negative restrictions of a general LPP.

2.5.2. Slack and Surplus Variables

In general, if any linear programming problem, we have a constraint of the type

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \text{ where } b_1 \geq 0$$

Then this inequality can be converted into an equation by adding one non – negative variable s_1 to the left-hand side. This new variable is called a slack variable and the constraints are transformed into the following equation:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 = b_1 \text{ where } s_1 \geq 0, b_1 \geq 0$$

Thus, a non – negative variable subtracted from the left – hand side of less than or equal to (\leq) type of a constraint that converts it into an equation is called a slack variable. The values of this

variable can be interpreted as the amount of unused resource.

Similarly, if in any linear programming problem, we have a constraint of the type

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1$$

Then this inequality can be converted into an equation by subtracting one non-negative variable s_1 from the left-hand side. This new variable is called a surplus variable. The value of this variable can be interpreted as the amount over and above the required minimum level.

Canonical Form

The characteristics of the canonical form are:

- Objective function should be of maximization form. If it is given in minimization form, it should be converted into maximization form.
- All the constraints should be of “ \leq ” type, except for non-negative restrictions. Inequality of “ \geq ” type, if any, should be changed to an inequality of the “ \leq ” type.
- All variables should be non-negative. If a given variable is unrestricted in sign (i.e., positive, negative or zero), it can be written as a difference of two non-negative variables. Suppose x is unrestricted in sign, then x can be written as $x = x' - x''$ where $x' \geq 0, x'' \geq 0$.

2.5.3. Example. Express the following LPP in Canonical form:

$$\text{Minimize } Z = 2x_1 + x_2 + 4x_3$$

Subject to the constraints:

$$-2x_1 + 4x_2 \leq 4$$

$$x_1 + 2x_2 + x_3 \geq 5$$

$$2x_1 + 3x_3 \leq 2$$

$x_1, x_2 \geq 0$ and x_3 is unrestricted in sign

Solution: Here, the objective function is of the minimisation form. We rewrite it in the maximisation form as follows:

$$\text{Minimise } Z = 2x_1 + x_2 + 4x_3$$

Thus, we have to maximise $-Z = -2x_1 - x_2 - 4x_3$. So the problem becomes,

$$\text{Maximise } Z' = -2x_1 - x_2 - 4x_3. \text{ where } Z' = -Z$$

Now the second constraint is of the type “ \geq ”. Hence to convert it into type “ \leq ”, we multiply the inequality by -1 and write

$$-x_1 - 2x_2 - x_3 \leq -5$$

Other constraints are already in the desired form. But x_3 is unrestricted in sign. So, we write

$$x_3 = x'_3 - x''_3, \text{ where } x'_3 \geq 0, x''_3 \geq 0.$$

The canonical form of the given problem, therefore, is

Maximise $Z' = -2x_1 - x_2 - 4(x'_3 - x''_3)$, where $Z' = -Z$

Subject to the constraints:

$$-2x_1 + 4x_2 \leq 4$$

$$-x_1 - 2x_2 - (x'_3 - x''_3) \leq -5$$

$$2x_1 + 3(x'_3 - x''_3) \leq 2$$

$$x_1 \geq 0, x_2 \geq 0, x'_3 \geq 0, x''_3 \geq 0$$

Standard Form:

The characteristics of the Standard Form are:

- 1) The objective function should be in the maximization form as we explained in canonical form.
- 2) The right-side element of each constraint should be non- negative. If it is negative, we multiply the inequality by -1.
- 3) All constraints should be expressed in the form of equations, except for the non- negative restrictions by augmenting slack or surplus variables.

2.5.4. Example. Express the following LPP in the standard form:

$$\text{Minimise } Z = 2x_1 + x_2 + 4x_3$$

subject to the constraints:

$$-2x_1 + 4x_2 \leq 4$$

$$x_1 + 2x_2 + x_3 \geq 5$$

$$2x_1 + 3x_3 \leq -2$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Solution: The objective function should be of maximization form, i.e. we have to

$$\text{Maximise } Z' = -2x_1 - x_2 - 4x_3, \text{ where } Z' = -Z$$

subject to the constraints:

$$-2x_1 + 4x_2 \leq 4$$

$$x_1 + 2x_2 + x_3 \geq 5$$

$$-2x_1 - 3x_3 \leq 2 \text{ [} \because \text{ Right side should be non- negative]}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Now the inequalities are to be converted to equations. Note that the first and third inequalities are of the type “less than or equal to (\leq)”.

Therefore, a slack variable is to be added to the left side of each of these inequalities. The second inequality is of the type “more than or equal to (\geq)”. So, a surplus variable is to be subtracted from the left side of this inequality.

Thus, a standard form of the given LPP is

$$\text{Max. } Z' = -2x_1 - x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3, \text{ where } Z' = -Z$$

subject to the constraints:

$$-2x_1 + 4x_2 + s_1 = 4$$

$$x_1 + 2x_2 + x_3 - s_2 = 5$$

$$-2x_1 - 3x_3 + s_3 = 2$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0.$$

2.5.5. Exercise.

1. Express the following LPP in

(i) Canonical form

(ii) Standard Form

$$\text{Minimise } Z = x_1 - 2x_2 + x_3$$

subjects to the constraints:

$$2x_1 + 3x_2 + 4x_3 \geq -4$$

$$3x_1 + 5x_2 + 2x_3 \geq 7$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

2. Max. $Z = 5x + 7y$

subjects to the constraints:

$$12x + 12y \leq 840 \Rightarrow x + y \leq 70$$

$$3x + 6y \leq 300 \Rightarrow x + 2y \leq 100$$

$$8x + 4y \leq 480 \Rightarrow 2x + y \leq 120$$

$$x \geq 0, y \geq 0$$

Answer.

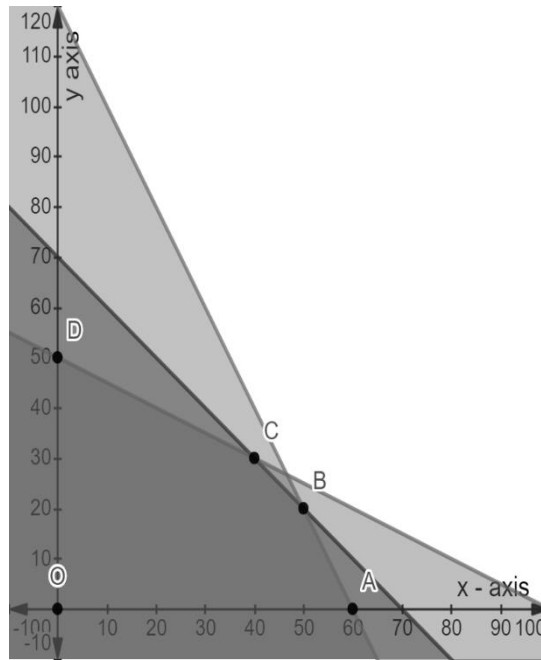


Fig.2.8.

The maximum value of Z is 410 at C(40,30) , i.e., at x = 40 , y = 30.

3. Minimise $z = 10x_1 + 4x_2$
 subjects to the constraints

$$4x_1 + x_2 \geq 80$$

$$2x_1 + x_2 \geq 60$$

$$x_1 \geq 0, x_2 \geq 0$$

Answer.

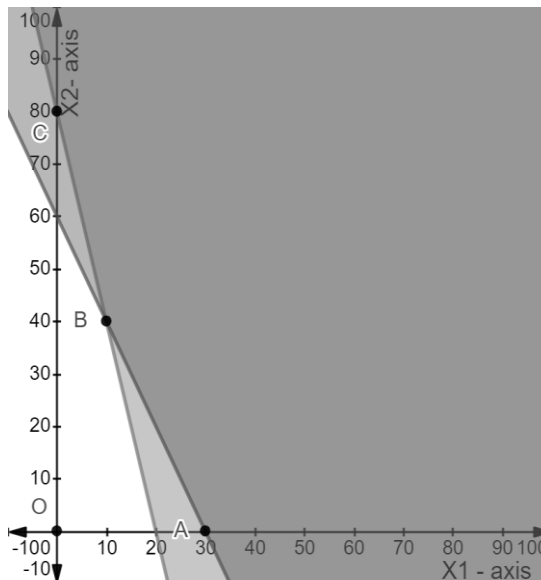


Fig.2.9.

The maximum value of Z is 260 at $B (10,40)$, i.e. $x_1 = 10$ and $x_2 = 40$.

Note: Had the objective function been of maximization form, the problem would have the unbounded solution. This is because the values of x_1 and x_2 could be increased beyond any limit, which would result in higher value of Z with no upper bound.

2.6. CHECK YOUR PROGRESS

1. What is linear programming? What are its major advantages and limitations?
2. What is meant by feasible solution of an LP problem?

2.7. SUMMARY

In this chapter, we introduced the concept of LPP and explain how these are formulated mathematically. We define objective function and graphical method of obtaining optimum value which is used to solve linear programming problems having two decision variables. Here, we study about feasible region/ solution set in detail. Also, we discuss canonical and standard form of an LPP as to solve LPP we need to convert them into a canonical or standard form.

3

SIMPLEX METHOD AND DUALITY IN LINEAR PROGRAMMING

Structure

- 3.1. Introduction
- 3.2. Simplex Method
- 3.3. Artificial Variable Techniques
- 3.4. Big-M Method
- 3.5. Two-Phase Method
- 3.6. Degeneracy
- 3.7. Duality in Linear Programming
- 3.8. Check Your Progress
- 3.9. Summary

3.1. INTRODUCTION

In chapter 2, we have studied the graphical method of solving linear programming problems and learnt how to express a linear programming problem in canonical and standard forms. As we know that the graphical method can be used only to solve the problems involving two decision variables and most of the real-life problems when mathematically formulated have more than two variables. For more than two decision variables, methods based on the concept of slack or surplus variables are used.

In this chapter, first we shall discuss the Simplex method for solving the linear programming problems involving more than two decision variables. After learning the procedure of simplex method, we discuss artificial variable techniques (Big-M Method and Two-Phase Method) for solving LPP and in last we shall discuss the concept of degeneracy in linear programming.

3.1.1. Objectives. The objective of these contents is to provide some important concepts/methods to the reader like:

- Simplex Method.
- Artificial Variable Techniques.
- Big-M and Two-Phase Method for Solving LPP Involving Artificial Variable(s).
- Degeneracy.

3.2. SIMPLEX METHOD

Simplex method was developed by G B Dantzig in 1947. The method is an iterative or step by step procedure by which one can obtained a new basic feasible solution from a given initial basic feasible solution. In this method, the value of the objective function improves with each solution and the optimum solution is achieved in a finite number of steps.

Suppose we have to optimize (maximise or minimise) Z , a linear function of n basic variables X_1, X_2, \dots, X_n . The LPP is written as:

$$\text{Maximise } Z = C_1X_1 + C_2X_2 + \dots + C_nX_n \quad \dots (1)$$

subject to the constraints:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ \cdot &\quad \cdot \quad \quad \quad \cdot \quad \cdot \quad (2) \\ \cdot &\quad \cdot \quad \quad \quad \cdot \quad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \end{aligned}$$

$$\text{and } x_j \geq 0 \text{ for all } j= 1,2, \dots, n \quad \dots (3)$$

where the constants C_1, C_2, \dots, C_n are the cost coefficients of decision variables. Let (a_{ij}) be $m \times n$ real matrix and $\{b_1, b_2, \dots, b_m\}$ be a set of constants.

The linear function Z gives in equation (1) is called the objective function. The set of inequalities gives in equation (2) is called constraints of LPP and the inequalities gives in equation (3) are known as non-negative restrictions of LPP (which means that all x_j values are non –negative)

Let us explain the step by step procedure for solving the LPP by the Simplex method.

Step 1: Convert the LPP into standard form by adding slack variables

We convert the given LPP into standard form by adding slack variables s_1, s_2, \dots, s_m .

$$\text{Maximise } Z = C_1X_1 + C_2X_2 + \dots + C_nX_n + 0s_1 + 0s_2 + \dots + 0s_m \quad \dots (4)$$

subject to the given constraints:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + s_2 &= b_2 \\ \cdot & \quad \cdot \quad \quad \quad \cdot \quad \quad \cdot \quad \quad \quad \dots (5) \\ \cdot & \quad \cdot \quad \quad \quad \cdot \quad \quad \cdot \quad \quad \quad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + s_m &= b_m \end{aligned}$$

$$\text{and } x_j \geq 0 \text{ and } s_i \geq 0 \text{ for all } i=1, 2, \dots, m \text{ and } j=1, 2, \dots, n \quad \dots (6)$$

Step 2: Construct the initial simplex table.

The initial simplex table is formed as follows:

$C_j \rightarrow$			C_1	C_2	$\dots C_r$	0	$0 \dots$	
Basic variables	Profit/unit (C_B)	Quantity	X_1	X_2	$\dots X_r$	S_1	$S_2 \dots$	Replacement Ratios
S_1	0	b_1	a_{11}	a_{12}	$\dots a_{1r}$	1	0..	
S_2	0	b_2	a_{21}	a_{22}	$\dots a_{2r}$	0	1..	
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	
	$Z_j \rightarrow$		$C_B X_1$	$C_B X_2$	$\dots C_B X_r$	$C_B S_1$	$C_B S_2$	
	$c_j - Z_j \rightarrow$							

Table 1: Initial Simplex Table

Step 3: Test for optimality

Calculate the values of $c_j - z_j$, the nature of the solution could be any one of the following:

1. If all $c_j - z_j \leq 0$, the solution under test is optimal solution.
2. If at least one value of $c_j - z_j$ is positive and corresponding to the most positive $c_j - z_j$, all the elements of the column X_j are negative or zero, the solution under test is an unbounded solution.
3. Suppose at least one value of $c_j - z_j$ is positive. Suppose the most positive value is, say $c_3 - z_3$ and at least one entry in the column of X_3 is positive. Then the solution under test is not optimal.

Step 4: Select incoming variable to enter the basis

If the solution is not optimal then we look for most positive entry and it could be any of $(c_j - z_j)$. In this case, we proceed as follows to obtain the optimal solution:

1. Let X_r be the variable which corresponds to the most positive value of $c_j - z_j$. This variable is called the incoming variable.
2. The column to be entered is called the key or pivot column.

Step 5: Test for feasibility (variable to leave the basis)

After finding the incoming variable, we determine the outgoing variable. For this we proceed as follow:

1. We divide the value of the Quantity (Qty) column by the corresponding positive values in the column X_r . These ratios are called Replacement Ratios (RR). Note that we do not consider the negative values in the column of X_r for calculating RR. Then we select the minimum RR. The basic variable corresponding to this value of the RR is called the outgoing variable. It is called outgoing variable because it is removed (goes out) from the next simplex table. The row selected in this manner is called key or pivot row.
2. The element that lies at the intersection of the key row and key column is called the key element or leading element or pivot element.

Step 6: Finding the new solution

1. We convert the key element to unity by dividing all entries in the row by the key element itself.
2. In the next step, we would like that the values of all other elements in the column corresponding to key element are zero. For this we carry out suitable operations on each row using the row containing the key element.

Step 7: Repeat the procedure

Go to step 3 and repeat the procedure until all entries in the $c_j - z_j$ are either negative or zero i.e. we repeat the procedure until either an optimal solution is obtained or there is an indication of unbounded solution.

The following example illustrate the simplex method:

3.2.1. Example. Maximise $Z=3x_1 + 2x_2$

subject to the constraints:

$$x_1 + x_2 \leq 4$$

$$x_1 - x_2 \leq 2$$

$$x_1 \geq 0, x_2 \geq 0$$

Solution: Step 1: First we convert the given LPP into standard form by adding slack variables s_1 and s_2 .

$$\text{Maximize } Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$$

subject to the constraints:

$$x_1 + x_2 + s_1 = 4$$

$$x_1 - x_2 + s_2 = 2$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Step 2: We now construct the initial simplex table (Table 2: Initial Simplex Table)

$c_j \rightarrow$			3	2	0	0
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2
s_1	0	4	1	1	1	0
s_2	0	2	1	-1	0	1
	$Z=$	$Z_j \rightarrow$ $c_j - Z_j \rightarrow$				

Note: Columns corresponding to the basic variables in initial simplex tables in the simplex method form an identity matrix. For example, in Table 2, the basic variables are s_1 and s_2 and their coefficients in the constraints form identity matrix.

The value of Z in column of Table2 is obtained from the equation

$$Z = \sum c_j b_j, j=1, 2$$

The values of z_j 's are obtained from the equation

$$z_j = \sum_{i=1}^m c_i a_{ij}, \text{ where } m \text{ is the number of rows. In this example } m=2$$

Next, we obtain $c_j - z_j$, known as net-evaluations. These are 3,2,0,0.

Thus, the table takes the following form (Table3):

Table 3: Simplex Table

$c_j \rightarrow$			3	2	0	0
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2
s_1	0	4	1	1	1	0
s_2	0	2	1	-1	0	1
	$Z=0$	$Z_j \rightarrow$ $c_j - Z_j \rightarrow$	0	0	0	0
			3	2	0	0

Step 3: Test for optimality

Since all $c_j - z_j \geq 0$, therefore the current solution is not optimal.

Step 4: Now we select incoming and outgoing variable.

For this, we select the most positive value of $c_j - z_j$, which is 3 in this case.

This corresponds to the variable x_1 which becomes the incoming variable. We shall enter it as a basic variable in the next simplex table.

Step 5: One of the variables s_1, s_2 will now be the outgoing variable which would be replaced by x_1 . To find out which one of these variables (s_1 or s_2) is the outgoing variable, we determine the replacement ratio (RR). Recall that RR for any row is obtained by dividing the value of Quantity column by the corresponding value of the Incoming variable for that particular row. For example, consider the first row in Table 3. The value of Quantity in the first row is 4 and the value of the incoming variable x_1 in this row is 1.

Therefore, RR for the first row is $\frac{4}{1}$. For the second row, Quantity is 2 and value of x_1 is 1. Hence, RR is $\frac{2}{1}$. RR is shown in the last column of the next simplex table (Table4).

The outgoing variable is the variable for which RR minimum. In Table 4, RR in the second row is minimum. It corresponds to s_2 and hence s_2 is the outgoing variable.

The element which lies at the intersection of the column of incoming variable (x_1) and the row of the outgoing variable (s_2) is called the key element. Here it is 1 and is enclosed by the rectangle as shown in Table4.

		Incoming variable		Key element			
$c_j \rightarrow$		3	2	0	0		
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2	R.R.
s_1	0	4	↓	1	1	0	4/1=4
(outgoing Variable) s_2	0	2	1	-1	0	1	2/1=2
	$Z=0$	$z_j \rightarrow$	0	0	0	0	
		$c_j - z_j \rightarrow$	3	2	0	0	

The objective function $Z=0$ at $x_1 = 0$ and $x_2 = 0$ (since x_1 and x_2 do not appear in the column of basic variables, these are non-basic variables. The values of non-basic variables are taken as zero). This is the initial solution, which we shall improve.

Step 6: Now, we form the next simplex table to find the adjacent vertex, i.e., the improved solution. The steps for forming this table are explained below:

- a) The initial simplex table (Table 4) has revealed that x_1 is an incoming variable which will enter in place of s_2 , the outgoing variable. The cost coefficient of s_2 will also be replaced by the cost coefficient of x_1 in “Profit / Unit” column. In this case its value is 3. Therefore, now the simplex table takes the form of Table 5. Other entries of the table will be filled up as explained in point (b).
- b) Since x_1 has entered as a basic variable, the coefficients of x_1 along with s_1 should form an identity matrix; i.e., the column corresponding to x_1 should be $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus, we have to make the key element unity and the other element zero. Note that it is already unity in this case (Table 4). Had it been any number other than unity, we would have divided the row containing leading element by the leading element itself, excluding the elements of the column “Profit / Unit”. So, the table takes the form

Table 5: Simplex Table

$c_j \rightarrow$			3	2	0	0
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2
s_1	0	4	1	1	1	0
x_1	3	2	1	-1	0	1

- c) Now, we have to make the other element in the column of key element (x_1) zero. In this case, its value is 1 ($a_{11} = 1$). For this, we multiply the row of the key element (excluding profit / unit) by negative of the element a_{11} (in this case) and add it to the first row.

This row operation is shown below:

First row of Table 7 \rightarrow 4 1 1 1 0

Second row of Table 7 \rightarrow -2 -1 1 0 -1

(on multiplying by -1)

Sum of the rows	2	0	2	1	-1
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Thus, the sum of the rows is the new first row (excluding profit/ unit), which replaces the first row of Table 5. We get Table 6 as follows:

Table 6: Simplex Table

$c_j \rightarrow$			3	2	0	0
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2
s_1	0	2	0	2	1	-1
x_1	3	2	1	-1	0	1

Note: Now the matrix for the basic variables s_1 and x_1 in the Table 6 is the identity matrix.

d) Next, we calculate z_j and $c_j - z_j$. The resulting simplex table is given below:

Table 7: Simplex Table

$c_j \rightarrow$			3	2	0	0	
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2	RR
$\leftarrow s_1$	0	2	0	2	1	-1	$2/2=1 \leftarrow$
x_1	3	2	1	-1	0	1	-
	$Z=0 \times 2 + 3 \times 2 = 6$	$z_j \rightarrow$	$0 \times 0 + 3 \times 1 = 3$	$0 \times 2 + 3 \times (-1) = -3$	$0 \times 1 + 3 \times 0 = 0$	$0 \times (-1) + 3 \times 1 = 3$	
		$c_j - z_j \rightarrow$	0	5 ↑	0	-3	

Here $Z=6$ at $x_1 = 2$ (see the value in the Quantity column) and $x_2=0$ (x_2 being non-basic variable).

In Table 7, the incoming variable is x_2 corresponding to the most positive value of $c_j - z_j = 5$. The key element is 2. We find the RR by dividing the elements in the Quantity column by the corresponding elements in the column of the incoming variable x_2 and ignore the negative or zero values. So, in this case, there will be only one RR and that will be considered as minimum RR. This implies that s_1 is the outgoing variable. If none of the elements in the column of incoming variable is positive, then the given LPP has an unbounded solution and we will stop there.

Step 7: Repeat the procedure

1. For the next simplex table, x_2 will enter in place of s_1 as a basic variable and accordingly we shall write the cost coefficient of x_2 in the LPP as the value for the column of profit / unit, i.e., 2. The element 2 enclosed in a rectangle in Table 7 is the key element. So, we shall divide the row containing the key element by the key element itself, i.e., by 2, excluding the values in the column of profit / unit. Thus, we get Table 8:

Table 8: Simplex Table

$c_j \rightarrow$			3	2	0	0
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2
x_2	2	1	0	1	1/2	-1/2
x_1	3	2	1	-1	0	1

Now the coefficients of x_2 and x_1 have to form an identity matrix, i.e., the column corresponding to x_2 should be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We have already made the key element unity. Now, to make the second element, (i.e., -1) in its column as zero, we just add the first row corresponding to x_2 , to the second row of Table 8 excluding the values in the column of profit / unit as follows:

First row of Table 10→	1	0	1	1/2	- 1/2
Second row of Table 10→	2	1	-1	0	1
(on multiplying by 1)					
Sum of the rows	3	1	0	1/2	1/2

We write the sum as above as the second row, excluded profit / unit as shown in Table 9. Then we obtain Z, z_j , and $c_j - z_j$ as explained in Step 1(iii) and (iv) and also in Step 2(c). The complete resulting simplex table is given below:

Table 9: Simplex Table

$c_j \rightarrow$			3	2	0	0	
Basic variables	Profit / Unit	Quantity	x_1	x_2	s_1	s_2	RR
x_2	2	1	0	1	1/2	-1/2	
x_1	3	3	1	0	1/2	1/2	
	$Z=2 \times 1 + 3 \times 3 = 11$	$z_j \rightarrow$	$2 \times 0 + 3 \times 1 = 3$	$2 \times 1 + 3 \times 0 = 2$	$2 \times 1/2 + 3 \times 1/2 = 5/2$	$2 \times (-1/2) + 3 \times 1/2 = 1/2$	
		$c_j - z_j \rightarrow$	0	0 ↑	-5/2	-1/2	

Now, in Table 9 none of the net-evaluation, i.e., the values of $c_j - z_j$ are positive. Therefore, the optimum solution is attained for $x_1 = 3$ and $x_2 = 1$, the values of x_1 and x_2 in the Quantity column of Table 11.

At this stage, it is necessary to check whether any of the non-basic variables (other than those appearing in the first column, i.e., the column with caption “Basic Variables”, i.e., x_1 and x_2 in Table 9) has value 0 in the net-evaluation row. If “yes”, then the LPP has multiple optimum solutions. If “not”, then we stop here concluding that the unique solution. In this example, it is given by

Max. $Z= 11$ at $x_1 = 3$ and $x_2 = 1$ (see the values in the Quantity column)

Note: It should be noted that in the initial simplex table (Table 2), $c_j - z_j$ is same as c_j . Also $c_j - z_j$ corresponding to the column of unit matrix are always zero. So, there is no need to calculate them.

We discuss the case of multiple optimum solutions in the next example.

3.2.2. Example. Max. $Z= 6x+3y$

Subject to the constraints:

$$2x + 5y \leq 120$$

$$2x + y \leq 40$$

$$x \geq 0, y \geq 0$$

Solution: Rewriting the given LPP in the standard form, we have

$$\text{Max. } Z = 6x + 3y + 0s_1 + 0s_2$$

subject to the constraints:

$$2x + 5y + s_1 = 120$$

$$2x + y + s_2 = 40$$

$$x, y, s_1, s_2 \geq 0$$

We form the initial simplex table (Table1) as explained in Example 1:

Table 1: Initial Simplex Table

$c_j \rightarrow$			6	3	0	0	
Basic variables	Profit / Unit	Quantity	x	y	s_1	s_2	RR
s_1	0	120	2	5	1	0	120/2=60
$\leftarrow s_2$	0	40	2	1	0	1	40/2=20
	Z=0	$z_j \rightarrow$	0	0	0	0	
		$c_j - z_j \rightarrow$	6 ↑	3	0	0	

Note from Table 1 that $c_j - z_j=6$ is maximum for x and RR is minimum for s_2 . Therefore, x is the incoming variable and s_2 is the outgoing variable. Then in the second simplex table, x will enter in place of s_2 as a basic variable. Profit / unit will be written accordingly. The key element is 2 and enclosed by the rectangle in Table 1. To make the key element unity, we divide the second row in Table 1 by 2 excluding the values of the column of profit / unit. So, we get Table 2 as follows:

Table 2: Simplex Table

$c_j \rightarrow$			6	3	0	0
Basic variables	Profit / Unit	Quantity	x	y	s_1	s_2
s_1	0	120	2	5	1	0
X	6	20	1	1/2	0	1/2

Now, we have to make the other element in column of x zero so that the coefficients of s_1 and x form the identity matrix. So, we multiply the second row of Table 2 (excluding the elements in the column of profit /unit) by the negative of the element (a_{ij}) in the row of s_1 and column of x in Table 1, i.e., by -2 and then add it to the first row of the Table 2 as follows

First row of Table 2→	120	2	5	1	0
Second row of Table 2→	-40	-2	-1	0	-1
(on multiplying by -2)					
Sum of the rows	80	0	4	1	-1

We also calculate Z , z_j , and $c_j - z_j$. So, the resulting completed simplex table is as follows:

Table 3: Simplex Table

$c_j \rightarrow$			6	3	0	0	
Basic variables	Profit / Unit	Quantity	x	y	s_1	s_2	RR
s_1	0	80	0	4	1	-1	
X	6	20	1	1/2	0	1/2	
	Z=120	$z_j \rightarrow$	6	3	0	3	
		$c_j - z_j \rightarrow$	0	0	0	-3	

As none of the net-evaluations is positive, the optimum solution is attained. The optimum solution is Max. $Z = 120$ when $x = 20$ (see the value in the Quantity column),

$$Y = 0 \text{ (since } y \text{ is non-basic variable)}$$

But we also note that the non-basic variable y has zero value in its net-evaluation row in Table 3. Therefore, the given LPP has multiple optimal solutions. To find another optimal solution, let us find another vertex at which Max. $Z = 120$.

So, another simplex table has to be formed. Here, instead of selecting the column corresponding to the most positive element in the net-evaluation row, we select the column of non-basic variable which has zero in the net-evaluation row, i.e., the column of y shown in Table 4.

Table 4: Simplex Table

$c_j \rightarrow$			6	3	0	0	
Basic variables	Profit / Unit	Quantity	x	y	s_1	s_2	RR
$\leftarrow s_1$	0	80	0	4	1	-1	$80/4=20 \leftarrow$
X	6	20	1	1/2	0	1/2	$20/(1/2)=40$
	Z=120	$z_j \rightarrow$	6	3	0	3	
		$c_j - z_j \rightarrow$	0	0 ↑	0	-3	

Note that in Table 4, the key element is 4 and the minimum RR corresponds to s_1 . So, s_1 is the outgoing variable and y is the incoming variable. We form the next simplex table (Table 5) following the steps explained for forming Table 3. The resulting simplex table (Table 5) is given as follows:

Table 5: Simplex Table

$c_j \rightarrow$			6	3	0	0	
Basic variables	Profit / Unit	Quantity	x	y	s_1	s_2	RR
Y	3	20	0	1	1/4	-1/4	
X	6	10	1	0	-1/8	5/8	
	Z=120	$z_j \rightarrow$	6	3	0	3	
		$c_j - z_j \rightarrow$	0	0	0	-3	

You should verify all entries of Table 5 before studying further. Note that for forming Table 5, we have first divided the first row by the value of key element to make the key element unity.

Then we obtain the second row of Table 5 as follows:

Second row of Table 4 \rightarrow 20 1 1/2 0 1/2

First row of Table 5 \rightarrow -10 0 -1/2 -1/8 1/8

(on multiplying by -1/2)

Sum of the rows 10 1 0 -1/8 5/8

From the above simplex table (Table 5), we find that Max. $Z = 120$ at (10,20) also (see the value of $x = 10$, $y = 20$ in the Quantity column).

So, we have two vertices at which the maximum value of Z is the same, i.e., 120. So, the other solutions of the LPP are obtained as follows:

First ordinate of other solutions = $t \times$ (First ordinate of first /vertex)

$$+ (1-t) \times (\text{First ordinate of first / vertex})$$

Second ordinate of other solutions = $t \times$ (Second ordinate of first /vertex)

$$+ (1-t) \times (\text{Second ordinate of first / vertex})$$

So, the other solutions are given as

$$\text{First ordinate} = t \times 10 + (1 - t) \times 20 = 10t + 20 - 20t = 20 - 10t$$

$$\text{Second ordinate} = t \times 20 + (1 - t) \times 0 = 20t$$

The other solutions are (20-10t, 20t), $0 \leq t \leq 1$.

3.2.3. Exercises.

Solve the following LPPs by the Simplex method:

1. Maximise $Z = 2x_1 + 4x_2$

subject to the constraints:

$$x_1 + 2x_2 \leq 5$$

$$x_1 + x_2 \leq 4$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

Answer. Max. $Z = 10$ at the two vertices $(0, 5/2)$ and $(3, 1)$. Max. $Z = 10$ at many other points also which are given as $(3 - 3t, 1 + 3t/2)$, $0 \leq t \leq 1$.

2. Maximise $Z = 100x_1 + 60x_2 + 40x_3$

Subject to the constraints:

$$x_1 + x_2 + x_3 \leq 100$$

$$10x_1 + 4x_2 + 5x_3 \leq 500$$

$$x_1 + x_2 + 3x_3 \leq 150, \quad x_1, x_2, x_3 \geq 0$$

Answer. $Z = 22000/3$ at $x_1 = 100/3, x_2 = 200/3, x_3 = 0$

3.3. ARTIFICIAL VARIABLE TECHNIQUES

After converting the given LPP into standard form, sometimes we observe that some of the variables in the standard form are surplus variables and the corresponding column vectors do not provide unit vectors for the initial basis and hence the column which form unit (identity) matrix are missing in the initial simplex table. Such a situation is usually observed if the constraints equation(s) is (are) of the type “ \geq ”. In order to have unit vectors in the basis, we introduce new type of variable (s) known as artificial variable(s). These are added to act as basic variable. They have no physical meaning and are used only to initiate the solution so that the simplex procedure may be adopted as usual till the optimal solution is obtained. The artificial variables are eliminated from the simplex table as and when they become non-basic variables. This technique is called the artificial variable technique for solving LPP. There are two methods for solving LPPs involving artificial variables.

i) Big-M or Method of Penalties or Charne’s Method

ii) Two-Phase Method

3.4. BIG-M METHOD

Big-M Method is used for removing artificial variable(s) from the basis. It is also known as Penalty method or Charne’s Method. In this method, the objective function coefficients impose a huge and hence unacceptable penalty. In case of maximisation, the objective function is modified by adding $-MA_1$, where M is arbitrary large and A_1 is an artificial variable. If there are two artificial variables A_1 and A_2 , then $-MA_1 - MA_2$ is added to the objective function. Similar treatment is done for more artificial variables.

The logic behind taking the coefficient as $-M$ is that we should never get the net-evaluation positive in the column of the artificial variable, i.e., the artificial variable should not enter again as a basic variable. M is very big and hence adding $-MA_1$ is the penalty to the objective function. Hence this method is called penalty method. Though $-M$ is big penalty, it does not affect the objective function. This is because the value of artificial variable should come out to be zero so that $-MA_1$ becomes zero. If the artificial variable remains as a basic variable till the final simplex table, then its value in the quantity column should be zero for the solution of LPP to exist. Otherwise, if in the final simplex table, an artificial variable appears as a basic variable and is non zero, the LPP does not possess any feasible solution.

3.4.1. Example: Maximise $Z = x_1 + 2x_2$

Subject to the constraints:

$$x_1 - x_2 \geq 3$$

$$2x_1 + x_2 \leq 10$$

$$x_1 \geq 0, x_2 \geq 0$$

Solution: First we convert the given LPP in the standard form as follows:

$$\text{Max } Z = x_1 + 2x_2 + 0s_1 + 0s_2$$

Subject to the constraints

$$x_1 - x_2 - s_1 = 3$$

$$2x_1 + x_2 + s_2 = 10$$

$$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$$

Now, let us try to form simplex table as follows:

Table 1: Simplex Table

		$c_j \rightarrow$	1	2	0	0
Basic variable	Profit/unit	Qty	x_1	x_2	s_1	s_2
			1	-1	-1	0
			2	1	0	1

We cannot form the initial simplex table with the variables x_1, x_2, s_1, s_2 as there is only one variable s_2 that has a column of unit matrix. Therefore, one more variable, i.e., artificial variable A_1 (say) needs to be introduced in the first constraints to get another column of unit matrix. Its coefficient in the objective function will be taken as $-M$.

Thus, the objective function is

$$\text{Max. } Z = x_1 + 2x_2 + 0s_1 + 0s_2 - MA_1$$

Subject to the constraints

$$x_1 - x_2 - s_1 + A_1 = 3$$

$$2x_1 + x_2 + s_2 = 10$$

$$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$$

The initial simplex table, therefore, is as follows (Table 2: Initial Simplex table)

		c_j	1	2	0	0	-M	
Basic variable	Profit/unit	Qty	x_1	x_2	s_1	s_2	A_1	RR
$\leftarrow A_1$	-M	3	1	-1	-1	0	1	$3/1=3 \leftarrow$
s_2	0	10	2	1	0	1	0	$10/2=5$
	$Z = -3M$	$z_j \rightarrow$ $c_j - z_j \rightarrow$	-M 1+M \uparrow	M 2-M	M -M	0 0	-M 0	

Here, since M is big, 1+M is most positive and x_1 is the incoming variable. The least replacement ratio is (3/1) which corresponds to A_1 and hence it is the outgoing variable. Thus, the resulting simplex table is as follows:

Table 3: Simplex Table

		$c_j \rightarrow$	1	2	0	0	-M	
Basic variable	Profit/unit	Qty	x_1	x_2	s_1	s_2	A_1	RR
x_1	1	3	1	-1	-1	0	\times	
$\leftarrow s_2$	0	4	0	3	2	1	\times	$4/3 \leftarrow$
	$Z = 3$	$z_j \rightarrow$ $c_j - z_j \rightarrow$ $z_j \rightarrow$	1 0	-1 3	-1 1	0 0	\times	

We obtain the second row (excluding profit/unit) in table 3 as follows:

First row of the second simplex table (table3) $\rightarrow -6 \quad -2 \quad 2 \quad 2 \quad 0 \quad \times$

(After multiplying by -2)

Second row of first simplex table (table2) $\rightarrow 10 \quad 2 \quad 1 \quad 0 \quad 1 \quad \times$

second row of the third simplex table $\rightarrow 4 \quad 0 \quad 3 \quad 2 \quad 1 \quad \times$

Once the artificial variable is removed from the basic variable, there is no need to do any computational work for it.

Thus, we get the resulting simplex table.

Table 4: Simplex Table

		$c_j \rightarrow$	1	2	0	0	-M	
Basic variables	Profit/unit	Qty	x_1	x_2	s_1	s_2	A_1	
x_1	1	13/3	1	0	-1/3	1/3	×	
x_2	2	4/3	0	1	2/3	1/3	×	
	$Z=21/3$	$z_j \rightarrow$	1	2	1	1	×	
	$=7$	$c_j - z_j \rightarrow$	0	0	-1	-1	×	

The first row (excluding profit/unit) of Table4 is obtained as follows:

Second row of third simplex table (Table 4) $\rightarrow 4/3 \quad 0 \quad 1 \quad 2/3 \quad 1/3 \quad \times$

(After multiplying by 1)

First row of second simplex table $\rightarrow 3 \quad 1 \quad -1 \quad -1 \quad 0 \quad \times$

Sum of rows	\rightarrow	13/3	1	0	-1/3	1/3	×
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Since none of the net-evaluation is positive, the optimum solution is attained and is given by

Max. $Z=7$ at $x_1 = 13/3$ and $x_2 = 4/3$.

3.4.2. Exercises.

1. Minimize $Z = 4x_1 + 2x_2$

Subject to the constraints

$$3x_1 + x_2 \geq 27$$

$$x_1 + x_2 \geq 21$$

$$x_1 + 2x_2 \geq 30$$

$$x_1, x_2 \geq 0$$

Answer. Max. $Z' = -48$, i.e. Max. $-Z = -48$ i.e. Min. $Z = 48$ when $x_1 = 3, x_2 = 18$

2. Maximise $Z = x_1 + 2x_2$

Subject to the constraints:

$$x_1 + x_2 \leq 4$$

$$x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Answer. No feasible solution.

3. Maximise $Z = 10x_1 + 2x_2$

Subject to the constraints:

$$-x_1 + x_2 \leq 2$$

$$x_1 + x_2 \geq 4$$

$$x_1, x_2 \geq 0$$

Answer. Unbounded solution.

3.5.TWO-PHASE METHOD

The two-phase method provides an alternate procedure for removing artificial variables from the basis by which one can not only get initial basic feasible solution but also eliminate redundant equation existing among the constraints. It also terminates the iteration if a feasible solution of the problem is absent, for this the method is divided into two phases of iterations.

In the first phase, the process of eliminating artificial variable is performed so that we get a basic feasible solution of the LPP and the second phase is used to get the optimal solution. Since the process of finding the solution of an LPP is completed in two phases, this is called the two-phase method.

Remark: The basic feasible solution (if it exists) obtained at the end of phase I is used to start phase II.

Rules for applying two-phase method are as follows:

1. Assign a cost '-1' to each artificial variable and a cost '0' to all other variables (in place of their original cost) in the objective function.
2. Solve the auxiliary problem by simplex method until either of the following three properties arise:
 - i. Max. $Z^* < 0$ and at least one artificial vector appears in the optimum basis at a positive level. In this case not proceed to Phase II.
 - ii. Max. $Z^* = 0$ and at least one artificial vector appears in the optimum basis at zero level. In this case proceed to phase II.
 - iii. Max. $Z^* = 0$ and no artificial vector appears in the optimum basis. In this case also proceed to phase II.

The method is well explained by the following examples.

3.5.1. Example. Use two-phase simplex method to solve the problem:

$$\text{Min. } Z = x_1 - 2x_2 - 3x_3$$

Subject to the constraints:

$$-2x_1 + x_2 + 3x_3 = 2$$

$$2x_1 + 3x_2 + 4x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

Solution. Max. $Z = -x_1 + 2x_2 + 3x_3$

Subject to the constraints:

$$-2x_1 + x_2 + 3x_3 + a_1 = 2$$

$$2x_1 + 3x_2 + 4x_3 + a_2 = 1$$

$$x_1, x_2, x_3, a_1, a_2 \geq 0$$

Phase I: Auxiliary linear programming problem is

$$\text{Max. } Z^* = 0x_1 + 0x_2 + 0x_3 - 1a_1 - 1a_2$$

Subject to the constraints:

$$-2x_1 + x_2 + 3x_3 + a_1 = 2$$

$$2x_1 + 3x_2 + 4x_3 + a_2 = 1$$

$$x_1, x_2, x_3, a_1, a_2 \geq 0$$

		C_j	0	0	0	-1	-1	
Basic variable	C_B	X_B	x_1	x_2	x_3	A_1	A_2	Minimum Ratio
a_1	-1	2	-2	1	3	1	0	2/3
$\leftarrow a_2$	-1	1	2	3	4	0	1	1/4 \leftarrow
		$Z^* = -3$	0	-4	-7 \uparrow	0	0	$\leftarrow \Delta_j$
a_1	-1	5/4	-7/2	-5/4	0	1	-3/4	
x_3	0	1/4	1/4	3/4	1	0	1/4	
		$Z^* = -5/4$	7/4	5/4	0	0	7/4	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$ an optimum basic feasible solution to the auxiliary Linear Programming Problem has been attained. But at the same time Max. Z^* is negative.

Here, there is no need to enter phase-II.

Note: Here in place of $c_j - z_j$, we take $-(c_j - z_j)$ as Δ_j so corresponding changes imposed.

3.5.2. Example. Solve the problem:

$$\text{Min. } Z = \frac{15}{4}x_1 - 3x_2$$

Subject to the constraints:

$$3x_1 - x_2 - x_3 \geq 3$$

$$x_1 - x_2 + x_3 \geq 2$$

$$\text{And } x_1, x_2, x_3 \geq 0$$

Solution. Min. $Z = \frac{15}{4}x_1 - 3x_2$

Subject to the constraints:

$$3x_1 - x_2 - x_3 - s_1 + a_1 = 3$$

$$x_1 - x_2 + x_3 - s_2 + a_2 = 2$$

And $x_1, x_2, x_3, a_1, a_2, s_1, s_2 \geq 0$

Phase: I

Max. $Z^* = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 - 1a_1 - 1a_2$

Subject to the constraints:

$$3x_1 - x_2 - x_3 - s_1 + a_1 = 3$$

$$x_1 - x_2 + x_3 - s_2 + a_2 = 2$$

And $x_1, x_2, x_3, a_1, a_2, s_1, s_2 \geq 0$

$S_1, S_2 \rightarrow$ Surplus

$a_1, a_2 \rightarrow$ Artificial variables.

Construct the simplex table:

		$C_j \rightarrow$	0	0	0	0	0	-1	-1	
Basic variable	C_B	X_B	x_1	x_2	x_3	s_1	s_2	A_1	A_2	Minimum Ratio
$\leftarrow a_1$	-1	3	3	-1	-1	-1	0	1	0	$1 \leftarrow$
a_2	-1	2	1	-1	1	0	-1	0	1	2
		$Z^* = -5$	$-4 \uparrow$	2	0	1	1	0	0	$\leftarrow \Delta_j$
$\rightarrow x_1$	0	1	1	$-1/3$	$-1/3$	$-1/3$	0	$1/3$	0	
$\leftarrow a_2$	-1	1	0	$-2/3$	$4/3$	$1/3$	-1	$-1/3$	1	$3/4 \leftarrow$
		$Z^* = -1$	0	$2/3$	$-4/3 \uparrow$	$-1/3$	1	$4/3$	0	$\leftarrow \Delta_j$
x_1	0	$5/4$	1	$-1/2$	0	$-1/4$	$-1/4$	$1/4$	$1/4$	
$\rightarrow x_3$	0	$3/4$	0	$-1/2$	1	$1/4$	$-3/4$	$-1/4$	$3/4$	
		$Z^* = 0$	0	0	0	0	0	1	1	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$ and no artificial variable appears in the basis, an optimum solution to the auxiliary problems has been attained.

Phase II:

In this phase, we consider

$$\text{Max. } Z' = -\frac{15}{2}x_1 + 3x_2 + 0x_3 + 0s_1 + 0s_2$$

Now, apply simplex method in the usual manner:

		$C_j \rightarrow$	-15/2	3	0	0	0	
Basic variable	C_B	X_B	x_1	x_2	x_3	s_1	s_2	Minimum Ratio
x_1	-15/2	5/4	1	-1/2	0	-1/4	-1/4	
x_3	0	3/4	0	-1/2	1	1/4	3/4	
		$Z' = -75/8$	0	3/4	0	15/8	15/8	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution is:

$$x_1 = 5/4, x_2 = 0, x_3 = 3/4, \text{Min. } Z = 75/8$$

3.5.3. Example: Max. $Z = 5x_1 + 8x_2$

Subjected to the constraints

$$3x_1 + 2x_2 \geq 3$$

$$x_1 + 4x_2 \geq 4$$

$$x_1 + x_2 \leq 5$$

$$\text{And } x_1, x_2 \geq 0$$

Solution: Max. $Z = 5x_1 + 8x_2$

Subjected to the constraints

$$3x_1 + 2x_2 - s_1 + a_1 = 3$$

$$x_1 + 4x_2 - s_2 + a_2 = 4$$

$$x_1 + x_2 + s_3 = 5$$

$$\text{And } x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0$$

Phase: I

$$\text{Max. } Z = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 - 1a_1 - 1a_2$$

Subjected to the constraints

$$3x_1 + 2x_2 - s_1 + a_1 = 3$$

$$x_1 + 4x_2 - s_2 + a_2 = 4$$

$$x_1 + x_2 + s_3 = 5$$

$$\text{And } x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0$$

		$C_j \rightarrow$	0	0	-1	-1	0	
Basic variable	C_B	X_B	x_1	x_2	A_1	A_2	s_3	Minimum Ratio
a_1	-1	3	3	2	1	0	0	3/2
$\leftarrow a_2$	-1	4	1	4	0	1	0	1 \leftarrow
s_3	0	5	1	1	0	0	1	5
		$Z = -7$	-4	-6 \uparrow	0	0	0	$\leftarrow \Delta_j$
a_1	-1	1	5/2	0	1	-1/2	0	2/5 \leftarrow
$\rightarrow x_2$	0	1	1/4	1	0	1/2	0	4
s_3	0	4	3/4	0	0	-1/4	1	16/3
		$Z = -1$	-5/2 \uparrow	0	0	3/2	0	$\leftarrow \Delta_j$
$\rightarrow x_1$	0	2/5	1	0	2/5	-5	0	
x_2	0	9/10	0	1	-1/10	3/2	0	
s_3	0	27/10	0	0	-3/10	7/2	1	
		$Z = 0$	0	0	1	1	0	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$ and no artificial variable appears in the basis, an optimum solution to the auxiliary problem has been attained.

Phase: II

In this phase, we consider

Max. $Z = 5x_1 + 8x_2 + 0s_3$

		$C_j \rightarrow$	5	8	0	
Basic variable	C_B	X_B	x_1	x_2	s_3	Minimum Ratio
x_1	5	2/5	1	0	0	
x_2	8	9/10	0	1	0	
s_3	0	27/10	0	0	1	
			0	0	0	$\leftarrow \Delta_j \geq 0$

$x_1 = 2/5, x_2 = 9/10$ and Max. $Z = \frac{46}{4}$

3.5.4. Exercise:

1. Solve the problem:

$$\text{Max. } Z = 2x_1 + 3x_2 + 5x_3$$

Subject to the constraints

$$3x_1 + 10x_2 + 5x_3 \leq 15$$

$$x_1 + 2x_2 + x_3 \geq 4$$

$$33x_1 - 10x_2 + 9x_3 \leq 33$$

$$\text{And } x_1, x_2, x_3 \geq 0$$

Answer. There do not exist any feasible solution, because artificial variable is not removed.

2. Max. $Z = 5x_1 - 2x_2 + 3x_3$

Subject to the constraints

$$2x_1 + 2x_2 - x_3 \geq 2$$

$$3x_1 - 4x_2 \leq 3$$

$$x_2 + 3x_3 \leq 5 \quad \text{And } x_1, x_2, x_3 \geq 0$$

Answer. $x_1 = 23/3$, $x_2 = 5$, $x_3 = 0$, $\text{Max. } Z = 85/3$.

3. Solve the problem:

$$\text{Min. } Z = x_1 + x_2$$

Subject to the constraints

$$2x_1 + x_2 \geq 4$$

$$x_1 + 7x_2 \geq 7$$

$$\text{And } x_1, x_2 \geq 0$$

Answer. $x_1 = 21/18$, $x_2 = 10/18$, $\text{Max. } Z = 31/13$

4. Solve the problem:

$$\text{Max. } Z = 2x_1 + x_2 + 1/4 x_3$$

Subject to the constraints

$$4x_1 + 6x_2 + 3x_3 \leq 8$$

$$3x_1 - 6x_2 - 4x_3 \leq 1$$

$$2x_1 + 3x_2 - 5x_3 \geq 4$$

$$\text{And } x_1, x_2, x_3 \geq 0$$

Answer. $x_1 = 9/7$, $x_2 = 10/21$, $x_3 = 0$, $\text{Max. } Z = 64/21$

5. Solve the problem:

$$\text{Max. } Z = 2x_1 + x_2$$

Subject to the constraints

$$5x_1 + 10x_2 - 3x_3 = 8$$

$$x_1 + x_2 + x_4 = 1$$

$$\text{And } x_1, x_2, x_3, x_4 \geq 0$$

Answer. $x_1 = 0, x_2 = 4/5, x_3 = 0, x_4 = 1/5, \text{Max. } Z = 4/5$

3.6. DEGENERACY

The process of obtaining a degenerate basic feasible solution in an LPP is called degeneracy. Degeneracy is revealed when a basic feasible solution contains a smaller number of non-zero variables than the number of independent constraints when at the initial stage while forming the initial simplex table if the values of some basic variables are zero and if at any iteration there is a tie in the minimum replacement ratios (RR) i.e. RR is not unique. In case of tie, if we select any of the rows arbitrarily, it may be possible that the subsequent iterations may not produce improvements in the value of objective function. This concept is known as cycling. In this case, we have to go back and choose another row.

However, one can avoid the situation of cycling which arises due to degeneracy in LPP by adopting the following procedure:

Determine the non-negative ratios of the first column of the unit matrix (and not the quantity column) to the entries of the entering variable. Then choose the minimum of the values occurring at the places of tie. If we find tie again, then we compute the ratios of the second column of the unit matrix to the entry of the entering variable. We continue the process till the ratios do not break the tie.

3.6.1. EXAMPLE. Maximise $Z = 3x_1 + 9x_2$

Subject to the constraints:

$$x_1 + x_2 \leq 8$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solution. The standard form of the LPP is

$$\text{Maximise } Z = 3x_1 + 9x_2 + 0s_1 + 0s_2$$

Subject to the constraints:

$$x_1 + 4x_2 + s_1 = 8$$

$$x_1 + 2x_2 + s_2 = 4$$

$$x_1, x_2, s_1, s_2 \geq 0$$

The initial simplex table for the given LPP is as follows:

Table 1: Initial Simplex Table

		$c_j \rightarrow$	3	9	0	0	
Basic variable	Qty		x_1	x_2	s_1	s_2	R.R.
s_1	0	8	1	4	1	0	$8/4=2$
s_2	0	4	1	2	0	1	$4/2=2$
	$Z = 0$	$z_j \rightarrow$	0	0	0	0	
		$c_j - z_j \rightarrow$	3	9	0	0	
				↑			

Clearly, from Table 1, there is tie in the minimum replacement ratio as it is 2 in each of the two rows. Now, to avoid cycling, we should not select any row arbitrarily. We will proceed as follows:

We divide the non-negative ratios of the first column of identify matrix, i.e.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ by the entries of the entering variable, i.e. } \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

So, we have replacement ratios as $\begin{pmatrix} 1/4 \\ 0/2 \end{pmatrix}$, i.e. $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$. So here we have found the minimum RR to be 0 and the tie has been broken. Therefore, we select the second row. Thus, the completed initial simplex table is as follows:

Table 2: Complete Initial Simplex Table

	$c_j \rightarrow$		3	9	0	0	
Basic Variables	Profit/unit	Qty	x_1	x_2	s_1	s_2	R.R.
s_1	0	8	1	4	1	0	$2, \frac{1}{4}$
← s_2	0	4	1	2	0	1	$2, 0 \leftarrow$
	$Z = 0$	$z_j \rightarrow$	0	0	0	0	
		$c_j - z_j \rightarrow$	3	9	0	0	
				↑			

Now we apply simply our simplex method procedure to obtain the optimum value of the objective function. It will be Max. $Z = 18$ at $x_1 = 0, x_2 = 2$.

3.7. DUALITY IN LINEAR PROGRAMMING

The concept of duality is very important in linear programming theory as duality implies that each linear programming problem can be analyzed in two different ways but having equivalent solutions in other words every LPP has associated with another LPP where the original problem is called primal and the other one is called its dual. The optimal solution of the primal reveals information about the optimal solution of the dual and vice-versa. This fact is very useful as sometimes dual of an LPP is easier to solve than its primal.

Note: In general, either problem can be considered as primal and the remaining one its dual.

Remark: The original problem is usually referred to as primal.

3.7.1. Steps for Converting Any Primal into Its Dual

Step I: First convert the objective function to maximization form.

Step II: If a constraint has inequality sign \geq , then convert the inequality sign \geq into \leq on multiply both sides by -1.

Step III: If a constraint has an equality sign ($=$) then, it is replaced by two constraints, one constraint having inequality sign \geq and other one having \leq and hence corresponding to one equality, we obtain two inequalities.

Step IV: Every unrestricted variable is replaced by the difference of two non-negative variables.

In this way, we get the standard form of the given primal. Now the dual of the given problem is obtained by:

Step V: Transposing the rows and columns of constraints coefficients. Also transposing the coefficients (c_1, c_2, \dots, c_n) of the objective function and the right-side constants

(b_1, b_2, \dots, b_m). Changing the inequalities from \leq to \geq sign and minimizing the objective function instead of maximizing it.

Note: If we have an equality constraint in primal then its corresponding dual variables must be unrestricted in sign.

Result 1. The dual of the dual is primal.

Result 2. If either the primal or the dual problem has an unbounded solution, the other one has no feasible solution.

Result 3. If either the primal or the dual problem has a finite optimal solution, the other problem also has finite optimal solution. The optimal value of the objective functions of the two problems are equal.

The steps of obtaining dual of an LPP are explained with the help of an example

3.7.2. Example. Find the dual of the following LPP:

$$\text{Min. } Z = 2x_2 + 5x_3$$

Subject to the constraints:

$$x_1 + x_2 \geq 2$$

$$2x_1 + x_2 + 6x_3 \leq 6$$

$$x_1 - x_2 + 3x_3 = 4$$

$$\text{And } x_1, x_2, x_3 \geq 0$$

Solution: Step: I

Change the objective function of minimization to maximization.

i.e. Max. $Z^* = -2x_2 - 5x_3$ where $Z^* = -Z$

Step: II

The inequality $x_1 + x_2 \geq 2$ can be written as $-x_1 - x_2 \leq -2$

Step: III

The equation $x_1 - x_2 + 3x_3 = 4$ can be expressed as a pair of inequalities:

$$x_1 - x_2 + 3x_3 \leq 4$$

$$x_1 - x_2 + 3x_3 \geq 4$$

Or

$$-x_1 + x_2 - 3x_3 \leq -4$$

$$x_1 - x_2 + 3x_3 \leq 4$$

Step: IV

Thus, original problem now becomes of the standard primal forms:

$$Z^* = 0x_1 - 2x_2 - 5x_3$$

Subject to the constraints

$$-x_1 - x_2 \leq -2$$

$$2x_1 + x_2 + 6x_3 \leq 6$$

$$-x_1 + x_2 - 3x_3 \leq -4$$

$$x_1 - x_2 + 3x_3 \leq 4$$

$$\text{And } x_1, x_2, x_3 \geq 0$$

Step: V

Thus, the required dual is:

$$\text{Min. } Z^* = -2w_1 + 6w_2 + 4w_3 - 4w_4$$

Subject to the constraints

$$-w_1 + 2w_2 + w_3 - w_4 \geq 0$$

$$-w_1 + w_2 - w_3 + w_4 \geq -2$$

$$6w_2 + 3w_3 - 3w_4 \geq -5$$

$$\text{And } w_1, w_2, w_3, w_4 \geq 0$$

3.7.3. Example. Write the dual of the following primal:

$$\text{Minimum } Z = 3x_1 - 2x_2 + 4x_3$$

Subject to the constraints:

$$3x_1 + 5x_2 + 4x_3 \geq 7$$

$$6x_1 - x_2 + 3x_3 \geq 4$$

$$7x_1 + 2x_2 - 3x_3 \leq 10$$

$$x_1 - 2x_2 + 5x_3 \geq 3$$

$$4x_1 + 7x_2 - 2x_3 \geq 2$$

$$\text{And } x_1, x_2, x_3 \geq 0$$

Solution. The original primal can be written in the standard primal form as:

$$\text{Max. } Z^* = -3x_1 + 2x_2 - 4x_3$$

Subject to the constraints:

$$\begin{aligned} -3x_1 - 5x_2 - 4x_3 &\leq -7 \\ -6x_1 - x_2 - 3x_3 &\leq -4 \\ 7x_1 - 2x_2 - x_3 &\leq 10 \\ -x_1 + 2x_2 - 5x_3 &\leq -3 \\ -4x_1 - 7x_2 + 2x_3 &\leq -2 \\ \text{And } x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Dual primal becomes

$$\text{Min. } Z^* = -7w_1 - 4w_2 + 10w_3 - 3w_4 - 2w_5$$

Subject to the constraints

$$\begin{aligned} -3w_1 - 6w_2 + 7w_3 - w_4 - 4w_5 &\geq -3 \\ -5w_1 - w_2 - 2w_3 + 2w_4 - 7w_5 &\geq 2 \\ 4w_1 - 3w_2 - 3w_3 - 5w_4 + 2w_5 &\geq -4 \end{aligned}$$

$$\text{And } w_1, w_2, w_3, w_4, w_5 \geq 0$$

3.7.4. Example. Given the dual Linear Programming Problem

$$\text{Min. } Z = 2x_1 + 3x_2 + 4x_3$$

Subject to the constraints:

$$2x_1 + 3x_2 + 5x_3 \geq 2$$

$$3x_1 + x_2 + 7x_3 = 3$$

$$x_1 + 4x_2 + 6x_3 \leq 5$$

$$\text{And } x_1, x_2 \geq 0, x_3 \text{ is unrestricted.}$$

Solution. Since the variable x_3 is unrestricted in sign, the given LPP can be transform into standard primal form by substituting $x_3 = x'_3 - x''_3$ where $x'_3 \geq 0, x''_3 \geq 0$

\therefore Standard primal becomes:

$$\text{Max. } Z^* = -2x_1 - 3x_2 - 4(x'_3 - x''_3)$$

Subject to the constrain

$$\begin{aligned} -2x_1 - 3x_2 - 5(x'_3 - x''_3) &\leq -2 \\ 3x_1 + x_2 + 7(x'_3 - x''_3) &\leq 3 \\ -3x_1 - x_2 - 7(x'_3 - x''_3) &\leq -3 \\ x_1 + 4x_2 + 6(x'_3 - x''_3) &\leq 5 \end{aligned}$$

$$\text{And } x_1, x_2, x'_3, x''_3 \geq 0$$

∴ Required dual is

$$\text{Min. } Z^* = -2w_1 + 3(w_2' - x_3'') + 5w_3$$

Subject to the constraints

$$\begin{aligned} -2w_1 + 5(w_2' - x_2'') + w_3 &\geq -2 \\ -3w_1 + (w_2' - x_2'') + 4w_3 &\geq -3 \\ -5w_1 + 7(w_2' - x_2'') + 6w_3 &\geq -4 \\ 5w_1 - 7(w_2' - x_2'') - 6w_3 &\geq 4 \end{aligned}$$

$$\text{And } w_1, w_2', w_2'', w_3 \geq 0$$

Again, we write

$$\text{Min. } Z^* = -2w_1 + 3w_2 + 5w_3$$

Subject to the constraints

$$\begin{aligned} -2w_1 + 5w_2 + w_3 &\geq -2 \\ -3w_1 + w_2 + 4w_3 &\geq -3 \end{aligned}$$

$$5w_1 - 7w_2 - 6w_3 = 4 \quad \text{And } w_1, w_2 \geq 0 \text{ and } w_3 \text{ is unrestricted.}$$

3.7.5. Exercises.

1. Write the dual of the following Linear Programming Problem

$$\text{Max. } Z = 2x_1 + 3x_2 + x_3$$

Subject to the constraints:

$$4x_1 + 3x_2 + 4x_3 = 6$$

$$x_1 - 2x_2 + 35 = 4$$

$$\text{And } x_1, x_2, x_3 \geq 0$$

$$\text{Answer. Min. } Z = 6(w_1 - w_2) + 4(w_3 - w_4)$$

Subject to the constraint

$$4(w_1 - w_2) + (w_3 - w_4) \geq 2$$

$$3(w_1 - w_2) + 2(w_3 - w_4) \geq 3$$

$$(w_1 - w_2) + 5(w_3 - w_4) \geq 1$$

$$\text{And } w_1, w_2, w_3 \geq 0$$

2. Write the dual of the following Linear Programming Problem

$$\text{Max. } Z = 3x_1 - 2x_2$$

Subject to the constraints:

$$x_1 \leq 4$$

$$x_2 \leq 6$$

$$x_1 + x_2 \leq 5$$

$$-x_2 \leq -1$$

$$\text{And } x_1, x_2, x_3 \geq 0$$

Answer. Min. $Z^* = 4w_1 + 6w_2 + 5w_3 - 1w_4$

Subject to the constraints:

$$w_1 + w_3 \geq 3$$

$$w_2 + w_3 - w_4 \geq -2 \quad \text{And } w_1, w_2, w_3, w_4 \geq 0$$

3. Min. $Z = x_1 + x_2 + x_3$

Subject to the constraints:

$$x_1 - 3x_2 + 4x_3 = 5$$

$$x_1 - 2x_2 \leq 3$$

$$2x_1 - x_3 \geq 4$$

And $x_1, x_2, \geq 0$, x_3 is unrestricted

Answer. Min. $Z^* = 5w_1 + 3w_2 - 4w_3$

Subject to the constraints

$$w_1 + w_2 - 2w_3 \geq -1$$

$$-3w_1 - 2w_2 \geq -1$$

$$4w_1 + 0w_2 - w_3 \geq -1$$

$$-4w_1 + 0w_2 + w_3 \geq 1 \quad \text{And } w_2, w_3 \geq 0 \text{ and } w_3 \text{ is unrestricted.}$$

4. Max. $Z = 3x_1 + x_2 + x_3 - x_4$

Subject to the constraints:

$$x_1 - 5x_2 + 3x_3 + 4x_4 \leq 5$$

$$x_1 + x_2 = -1$$

$$x_3 - x_4 \geq -5$$

And $x_1, x_2, x_3, x_4 \geq 0$

Answer. Min. $Z^* = 5w_1 - w_2 + 5w_3$

Subject to the constraints

$$w_1 + w_2 \geq 3$$

$$5w_1 + w_2 \geq 1$$

$$3w_1 - w_3 \geq 1$$

$$4w_1 + w_3 \geq -1$$

And $w_1, w_3 \geq 0$ and w_2 is unrestricted.

3.8. CHECK FOR PROGRESS

1. Describe simplex method of solving linear programming problem.
2. Explain the term ‘Artificial variable’ and its use in linear programming.
3. What do you mean by two-phase method in linear programming problem?
4. Write a note on the use of Big-M method in solving linear programming problem.
5. Explain the concept of degeneracy in simplex method.
6. Explain the concept of duality.
7. Discuss relationship between primal and its dual.

3.9. SUMMARY

In this chapter, we study simplex method which can be used for solving LPP involving more than two variables. Simplex method is an iterative method as it examines the extreme points in a systematic manner, repeating the same set of steps until an optimal solution is reached. We have introduced artificial variables which play an important role to solve an LPP involving greater-than-or-equal to constraints and/or equality constraints.

Here, we discuss two important methods namely Big-M and Two-phase method for solving LPP involving artificial variables. In this chapter, we introduce the concept of degeneracy which occurs in a linear programming problem when in the simplex table one or more basic variable has value zero or RR is not unique. In the last, we explain duality concept of LPPs which is very useful in science and engineering, game theory, economics etc.

4

TRANSPORTATION PROBLEM

Structure

- 4.1. Introduction
- 4.2. Mathematical Formulation of the Transportation Problem
- 4.3. Methods of Finding Initial Basic Feasible Solution
- 4.4. Methods of Finding Optimal Solution
- 4.5. Unbalanced Transportation Problem
- 4.6. Degeneracy
- 4.7. Check Your Progress
- 4.8. Summary

4.1. INTRODUCTION

Transportation problem is a special kind of Linear Programming Problem (LPP) in which the objective is to transport goods from a set of sources/origins to a set of destinations in such a manner that the total transportation or shipping cost is minimized. To achieve this objective, we must know about some parameters such as the quantity of available supplies, the quantity demanded and the costs of shipping a unit from various origins to various destinations. Solving LPPs by simplex method discussed in chapter 3 involves a large number of variables and constraints and takes a long time to solve it. So, in this chapter, we shall discuss the methods, which are specifically applied for solving transportation problems. Firstly, we shall explain how to formulate mathematically a transportation problem and methods of finding initial basic feasible solution. After finding the initial basic feasible solution, we discuss how optimality test is

performed by applying the Stepping Stone Method or Modified Distribution Method (MODI) to find whether the obtained feasible solution is optimal or not. In the last, we shall explain the case of unbalanced transportation problems and degeneracy.

4.1.1. Objectives

After studying this chapter, the reader should be able to:

- Define a transportation problem.
- Obtain the basic feasible solution of a transportation problem using North – West Corner Rule, Least Cost and Vogel’s Approximation methods.
- State the conditions for performing optimality test.
- Explain the algorithm of the Stepping Stone and Modified Distribution (MODI) methods of obtaining the optimal solution of a transportation problem.
- Solve the transportation problems for special cases such as unbalanced transportation problem, case of degeneracy.

4.2. MATHEMATICAL FORMULATION OF THE TRANSPORTATION PROBLEM

Let there be m origins/ sources of supply $O_1, O_2, \dots, O_i \dots O_m$ and n destinations $D_1, D_2, \dots, D_j \dots D_n$. The total number of the capacities of all m origins is assumed to be equal to the total number of the requirements of all n destinations. Let C_{ij} be the cost of shipping one unit from origin i to destination j. Let a_i be the capacity/ availability of items at origin i and b_j , the requirement/demand of the destination j. Then this transportation problem can be expressed in a tabular form as follows:

Origin	Destinations						Availability/ capacity
	D_1	D_2	...	D_j	...	D_n	
O_1	C_{11}	C_{12}	...	C_{1j}	...	C_{1n}	a_1
O_2	C_{21}	C_{22}	...	C_{2j}	...	C_{2n}	a_2
\vdots	\vdots	\vdots		\vdots		\vdots	\vdots
O_i	C_{i1}	C_{i2}	...	C_{ij}	...	C_{in}	a_i
\vdots	\vdots	\vdots		\vdots		\vdots	\vdots
O_m	C_{m1}	C_{m2}	...	C_{mj}	...	C_{mn}	a_m
Requirement/ Demand	b_1	b_2	...	b_j	...	b_n	Total

The condition for the existence of a feasible solution to a transportation problem is give as

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

The above equation tells us that the total requirement/demand equals the total capacity. If it is not so, a dummy origin or destination is created to balance the total capacity and requirement.

Now let x_{ij} be the number of units to be transported from origin i to destination j and C_{ij} the corresponding cost of transportation. Then the total transportation cost is $\sum_{i=1}^m \sum_{j=1}^n C_{ij}x_{ij}$

Subject to the constraints:

$$\sum_{j=1}^n x_{1j} = a_1, \sum_{j=1}^n x_{2j} = a_2, \dots, \sum_{j=1}^n x_{mj} = a_m$$

.....(3)

$$\sum_{i=1}^m x_{i1} = b_1, \sum_{i=1}^m x_{i2} = b_2, \dots, \sum_{i=1}^m x_{in} = b_n$$

And $x_{ij} \geq 0$ for all $i = 1, 2, 3 \dots m$ and $j = 1, 2 \dots n$.

The Simplex method is regarded as the most generalized method to solve this. However, the solution is very lengthy and takes a long time to solve it since a large number of decision variables and artificial variables are involved. It is far simpler to solve it by transportation method as compared to the Simplex method. In the transportation method, we first obtain the initial basic feasible solution and then perform the optimality test.

Note: A transportation problem is said to be balanced if it satisfies the condition $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$.

4.3. METHODS OF FINDING INITIAL BASIC FEASIBLE SOLUTION

There are several methods to obtain initial basic feasible solution. Here, we shall discuss the following methods to determine the initial basic feasible solution:

- (i) **North-West Corner Rule**
- (ii) **Least Cost Method**
- (iii) **Vogel's Approximation Method (Penalty or Regret Method)**

Vogel's Approximation method generally gives a solution closer to the optimum solution. Hence, it is preferred to the other two methods.

4.3.1. North – West Corner Rule

The North – West Corner Rule (NWC) is a simple and efficient method to obtain initial basic feasible solution. It can be summarized as follows:

Step 1: Start with cell (1, 1) at the north-west corner (upper left-hand corner) of the transportation matrix and allocate as much as possible there.

Step 2: Here, we have three cases

- (a) If the quantity needed at First Destination (b_1) is less than the quantity available at First Origin (a_1), we allocate a quantity equal to the requirement at First Destination to the cell (1, 1). At

this stage, Column 1 is exhausted, so we cross it out. Since the requirement b_1 is fulfilled, we reduce the availability a_1 by b_1 and proceed to north-west corner of the resulting matrix, i.e., cell (1,2).

- (b) If the quantity needed at First Destination (b_1) is greater than the quantity available at First Origin(a_1), allocate a quantity equal to the quantity available at First Plant/Origin (a_1) to cell (1, 1). At this stage, Row 1 is exhausted, so we cross it out and proceed to north-west corner of the resulting matrix, i.e., cell (2, 1).
- (c) If the quantity needed at First Destination (b_1) is equal to the quantity available at First Origin (a_1), we allocate a quantity equal to the requirement at First Destination (or the quantity available at First Origin). At this stage, both column 1 as well as Row 1 is exhausted. We cross them out and proceed to the north-west corner of the resulting matrix, i.e., cell (2, 2).

Step 3: We continue the procedure, until we reach the south – east corner of the original matrix.

4.3.1.1. Example. Find the basic feasible solution of the given transportation problem by applying North – West Corner rule:

Warehouse Factory	D	E	F	G	Capacity
A	42	48	38	37	160
B	40	49	52	51	150
C	39	38	40	43	190
Requirement	80	90	110	220	500

Solution. We start from the North – West corner, i.e., the Factory A and Warehouse D. The quantity needed at the First Warehouse (Warehouse D) is 80, which is less than the quantity available (160) at the First Factory A. Therefore, a quantity equal to the warehouse D is to be allocated to the cell (A, D). Thus, the requirement of Warehouse D is met by Factory A. So, we cross out column 1 and reduce the capacity of Factory A by 80. Then we go to cell (A, E), which is North – West corner of the resulting matrix.

Now, the quantity needed at the second Warehouse (Warehouse E) is 90, which is greater than the quantity available (80) at the First Factory A. Therefore, we allocate a quantity equal to the capacity at Factory A, i.e., 80 to the cell (A, E). The requirement of Warehouse E is reduced to 10. The capacity of Factory A is exhausted and has to be removed from the matrix. Therefore, we cross out row 1 and proceed to cell (B, E).

Now, the quantity needed at the second Warehouse (Warehouse E) is 10, which is less than the quantity available at the Second Factory B, which is 150. Therefore, the quantity 10 equal to the requirement at Warehouse E is allocated to the cell (B, E). Hence, the requirement of Warehouse E is met and we cross out column 1. We reduce the capacity of Factory B by 10 and proceed to cell (B, F).

Again, the quantity needed at the Third Warehouse (Warehouse F) is 110. It is less than the quantity available at the Second Factory (Factory B), which is 140. Therefore, a quantity equal to the requirement at Warehouse F is allocated to the cell (B, F). Since the requirement of Warehouse F is met, we cross out Column 1 and reduce the capacity of Factory B by 110. Then we proceed to cell (B, G). Now, the quantity needed at the Fourth Warehouse (Warehouse G) is 220, which is greater than the quantity available at the Second Factory (Factory B). Therefore, we allocate the quantity equal to the capacity of Factory B to the cell (B, G) so that the capacity of Factory B is exhausted and the requirement of Warehouse G is reduced to 190. Hence, we cross out Row 1 and proceed to cell (C, G).

Thus, the allocations given using North – West corner rule are as shown in the following matrix along with the cost per unit of transportation:

Warehouse \ Factory	D	E	F	G	Capacity
A	42 80	48 80	38	37	160
B	40	49 10	52 11	51 30	150
C	39	38	40	43 190	190
Requirement	80	90	110	220	500

Thus, the total transportation cost for these allocations

$$= 42 \times 80 + 48 \times 80 + 49 \times 10 + 52 \times 110 + 51 \times 30 + 43 \times 190$$

$$= 3360 + 3840 + 490 + 5720 + 1530 + 8170 = 23110$$

4.3.1.2. Exercise. Find the basic feasible solution of the following problem using North-West Corner Rule:

Origin/ Distribution Centre	1	2	3	4	5	6	Availability
1	4	6	9	2	7	8	10
2	3	5	4	8	10	0	12
3	2	6	9	8	4	13	4
4	4	4	5	9	3	6	18
5	9	8	7	3	2	14	20
Requirements	8	8	16	3	8	21	

Answer. Using North - West corner rule, the allocations are to be made as under:

8 units to cells (1,1), 2 units to cell (1,2), 6 units to cell (2,2), 6 units to cell (2,3), 4 units to cell (3,3), 6 units to cell (4,3), 3 units to cell (4,4), 8 units to cell (4,5), 1 unit to cell (4,6) and 20 units to cell (5,6) and the transportation cost is equals to 501.

4.3.2. Least Cost Method

This method is also known as the Matrix Minimum method or Inspection method. It starts by making the first allocation to the cell for which the transportation cost per unit is lowest. The row or column for which the capacity is exhausted or requirement is satisfied is removed from the transportation table. We follow the procedure with the reduced matrix until all the requirements are satisfied. If there is a tie for the lowest cost cell while making any allocation, the choice may be made for a row or a column by which maximum requirement is exhausted. If there is a tie in making this allocation as well, then we can arbitrarily choose a cell for allocation.

The method can be easily explained with the help of the following example.

4.3.2.1. Example. Find the basic feasible solution of the transportation problem of Example 4.3.1.1. by using the Least Cost method.

Solution. Here, the least cost is 37 in the cell (A, G). The requirement of the Warehouse G is 220 and the capacity of Factory A is 160. Hence, the maximum number of units that can be allocated to this cell is 160. Thus, Factory A is exhausted. The requirement of Warehouse G is reduced by 160.

Now, the least cost is 38, which is in the cell (C, E). The requirement of the Warehouse E is 90 and the capacity of Factory C is 190. Hence, the maximum number of units that can be allocated to this cell is 90. Moreover, we reduce the capacity of factory C by 90.

The least cost in the matrix is 39, which is in the cell (C, D). The requirement of Warehouse D is 80 and the capacity of Factory C is 100. Hence, the maximum number of units that can be allocated to this cell is 80. The requirement of Warehouse D is exhausted. The capacity of the Factory C is also reduced by 80.

The least cost in this matrix is 40 which is in the cell (C, F). The requirement of Warehouse F is 110 and the capacity of Factory C is 20. Hence, the maximum number of units that can be allocated to this cell is 20. Thus, Factory C is exhausted. The requirement of Warehouse F is reduced by 20. It is now 90 in the reduced matrix.

The least cost is 51 in the cell (B, G) and the requirement of warehouse G is 60 units. So, we allocate 60 units to cell (B, G) and the remaining 90 units to the cell (B, F). Thus, the allocations given using Least Cost method are as shown in the following matrix along with the cost per unit of transportation:

Warehouse \ Factory	D	E	F	G	Capacity
A	42	48	38	37	160
B	40	49	52	51	60
C	39	38	40	43	190
Requirement	80	90	110	220	500

$$\begin{aligned} \text{Thus, the total transportation cost} &= 37 \times 160 + 52 \times 90 + 51 \times 80 + 38 \times 90 + 40 \times 20 \\ &= 21000 \end{aligned}$$

Note: This method has reduced the total transportation cost in comparison to the NWC rule.

4.3.2.2. Exercise. Find the basic feasible solution of the following problem using the Least Cost method:

Origin/ Distribution Centre	1	2	3	4	5	6	Availability
1	4	6	9	2	7	8	10
2	3	5	4	8	10	0	12
3	2	6	9	8	4	13	4
4	4	4	5	9	3	6	18
5	9	8	7	3	2	14	20
Requirements	8	8	16	3	8	21	

Answer. Using Least Cost method, the allocations are to be made as under:

4 units to cell (1,1), 3 units to cell (1,4), 3 units to cell (1,6), 12 units to cell (2,6), 4 units to cell (3,1), 8 units to cell (4,2), 10 units to cell (4,3), 6 units to cell (5,3), 8 units to cell (5,5) and 6 units to cell (5,6).
The transportation cost = 278.

4.3.3. Vogel's Approximation Method (VAM)

We describe the step by step procedure for finding the initial basic feasible solution by Vogel's Approximation method (Penalty method) in the following steps:

- (i) In the transportation table calculate penalties for each row (column), by taking the difference between the least and second least costs in the same row (column). We display it to the right (below) of that row (column) in a new column (row) formed by extending the table on the right (bottom). The new column and row formed by extending the table at the right and bottom are labelled as penalty column and penalty row, respectively. The differences noted in the penalty row or penalty column indicates the penalty or extra cost. If two cells in a row (or column) contain the same least costs then the difference is taken as zero.
- (ii) Select the row or column with the largest penalty (largest difference) and allocate the maximum possible units to the least cost cell in the selected column or row. If there is a tie in the values of penalties, the choice may be made for that row or column, which has the least cost. In case there is a tie in such least cost as well, choice may be made from that there is a tie in such least cost as well, choice may be made from that row or column by which maximum requirements are exhausted. The cell so chosen is allocated the units and the corresponding exhausted row or column is removed or ignored from further consideration.
- (iii) Now, we determine the column and row differences for the reduced transportation table and repeat the procedure until all column and row totals are exhausted.

This method is also known as the penalty method. Let us understand the procedure with the help of an example.

4.3.3.1. Example. Apply the Vogel's Approximation Method for finding the Basic Feasible Solution for the transportation problem of Example 4.3.1.1.

Solution. In the first row, the least and the second least costs are 37 and 38 and their difference is 1. We write 1 in a new column created on the right. It is labelled Penalty. Similarly, the differences between the least and the second least costs in the second and third row, respectively, are $49 - 40 = 9$ and $39 - 38 = 1$. So, we write the values (differences), i.e., 9 and 1 in the penalty column.

Next, we find the differences of the least and second least elements of each of the columns D, E, F and G. These are $40 - 39 = 1$, $48 - 38 = 10$, $40 - 38 = 2$ and $43 - 37 = 6$, respectively. We write them in a newly created penalty row at the bottom of the table.

We now select the largest of these differences in the penalty row and column, which are 10 in this case. This value (10) corresponds to the second column (Column E) and the least cost in the column is 38. Hence the allocation of 90 units (the maximum requirement of warehouse E) is to be made in the cell (C, E) from Factory C. Since the column corresponding to E is exhausted, it is removed for the next reduced matrix and the capacity of C is reduced by 90.

We now take the differences between the least and the Second least cost for each row and column of the reduced matrix. In the first row, the least and the second least costs are 37 and 38 and their differences is 1. We write it in the newly created penalty column. Similarly, we write the second difference element $51 - 40 = 11$ and third difference element $40 - 39 = 1$ in the second and third row of this column. Likewise, the differences of the smallest and second smallest elements of each of the columns D, F, and G are $40 - 39 = 1$, $4 - 38 = 2$ and $43 - 37 = 6$, respectively. We write these in a newly created penalty row at the bottom of the table.

Now, we select the largest of these differences in the penalty row and column, which is 11 in this case. This value (11) corresponds to Row B. Since the least cost in row is 40, we allocate 80 units (the maximum requirement of Warehouse D) to the cell (B, D). Thus, the requirement of Warehouse D is exhausted and we can remove it. We also reduce the capacity of Factory B by 80 in the next reduced matrix.

Again, in the first row, the least and the second least costs are 37 and 38 and their difference is 1. We write it to the right of this row in the newly created penalty column. Similarly, the second and third elements in the second and third rows of this column are $52 - 51 = 1$ and

$43 - 40 = 3$, respectively. We write these in a newly created penalty row at the bottom of the table. Now, we select the largest of these differences, which are 6 in this case. It corresponds to Column G and the least cost in this column is 37. Hence, we allocate 160 units (the maximum capacity of Factory A) to the cell (A, G). Since Row A is exhausted, it is removed for the next reduced matrix. We also reduce the requirement of Warehouse G by 160 units.

Once again, the difference of the least costs in the first row is $52 - 51 = 1$. We write it in the newly created penalty column. Similarly, for the second row, the difference is $43 - 40 = 3$. Likewise, the differences of the least and second least elements of each of the columns F and G are $52 - 40 = 12$ and $51 - 43 = 8$, respectively. We write them in the newly created penalty row at the bottom of the table. The largest of these differences is 12 in this case. It corresponds to Column F and the least cost in this column is 40. Hence, we allocate 100 units from the row (the maximum capacity of Factory C) to the cell (C, F). Since Row C is exhausted, it is removed and the requirement of Warehouse F is reduced to 10 for the next reduced matrix.

At the end, of the 70 units available in Factory B, we allocate 60 units to the lower cost (51), i.e., to the cell (B, G) and the remaining 10 units to the cell (B, F).

The entire procedure of allocating units by Vogel’s Approximation Method is given in the following table:

Warehouse Factory	D	E	F	G	Capacity	Diff ₁	Diff ₂	Diff ₃	Diff ₄
A	42	48	38	37 ¹⁶⁰	160	1	1	1	-
B	40 ⁸⁰	49	52 ¹⁰	51 ⁶⁰	150	9	11*	1	1
C	39	38 ⁹⁰	40 ¹⁰⁰	43	190	1	1	3	3
Requirement	80	90	110	220	500				
Diff ₁	1	10*	2	6					
Diff ₂	1	-	2	6					
Diff ₃	-	-	2	6*					
Diff ₄	-	-	12*	8					

Thus, the total transportation cost

$$\begin{aligned}
 &= 40 \times 80 + 38 \times 90 + 52 \times 10 + 40 \times 100 + 37 \times 160 + 51 \times 60 \\
 &= 3200 + 3420 + 520 + 4000 + 5920 + 306 = 20120
 \end{aligned}$$

Note: The total transportation cost obtained above is the lowest total transportation cost among the three methods. Clearly the solution obtained by VAM is nearest to the optimal solution.

4.3.3.2. Exercise. Find the basic feasible solution of the following problem using Vogel’s Approximation method:

Origin /Distribution Centre	1	2	3	4	5	6	Availability
1	4	6	9	2	7	8	10
2	3	5	4	8	10	0	12
3	2	6	9	8	4	13	4
4	4	4	5	9	3	6	18
5	9	8	7	3	2	14	20
Requirement	8	8	16	3	8	21	

Answer. The total transportation cost= 242.

4.4. METHODS OF FINDING OPTIMAL SOLUTION

Once an initial basic feasible solution is determined, the next step is to check its optimality. For this, we performed an optimality test which tell us that whether the obtained feasible solution is optimal or not. This can be done by applying one of the following methods:

- (a) Stepping Stone Method
- (b) Modified Distribution (MODI) Methods

These methods not only tell us about optimality of the initial basic feasible solution, they also improve the solution until the optimal solution is obtained. Use of the Stepping Stone method is convenient to a problem of small dimension as its application to a problem of a large dimension is quite tedious and cumbersome.

The MODI method is usually preferred over the Stepping Stone method. Before studying the methods of performing optimality test and finding optimal solutions, we introduce the concept of independent allocations and state the conditions for performing optimality test.

4.4.1. Independent and Non-Independent Allocations

A set of allocations is said to be independent if they do not form a loop in the transportation table. Here, formation of closed loop means that it is possible to travel from any allocation, back to itself by a series of horizontal and vertical jumps from one occupied cell (i.e., the cell containing allocation) to another, without a direct reversal of route (see Fig.). If such a loop can be formed using some or all of the allocations under consideration, the allocations are known as non-independent.

	•		•
•	•	•	
			•
		•	

(a)

•	•		•
•	•		
	•		•

(b)

Clearly allocations in (a) are independent allocations and of (b) are dependent.

Note: Every loop has an even number of cells.

4.4.2. Test for Optimality

An optimality test can be applied to the feasible solution only if it satisfies the following conditions:

- (i) It contains exactly $m+n-1$ allocations where m and n represent the number of rows and columns, respectively, of the transportation table.
- (ii) These allocations are independent.

4.4.3. Stepping Stone Method

Steps involved in Stepping Stone method for obtaining the optimal solution of a transportation problem can be summarized as follows:

1. First find an initial basic feasible solution.
2. Check optimality conditions if conditions are satisfied then evaluate all unoccupied cells for the effect of transferring one unit from an occupied cell to the unoccupied cell as follows:
 - a) Select an unoccupied cell to be evaluated.
 - b) Starting from this cell, from a closed path (or loop) through at least three occupied cells. The direction of movement is immaterial because the result will be same in both directions. Note that expect for the evaluated cell, all cells at the corners of the loop have to be occupied.
 - c) At each corner of the closed path, assign plus (+) and minus (-) sign alternatively; beginning with the plus sign for the unoccupied cell to be occupied.
 - d) Compute the net change in cost with respect to the costs associated with each cell traced in the closed path.
 - e) Repeat steps 2(a) to 2(d) until the net change in cost has been calculated for all unoccupied cells.
3. If the net changes are positive or zero, an optimal solution has been arrived at. Otherwise go to step 4.
4. If some net changes are negative, select the unoccupied cell having the most negative net change. If two negative values are equal, select the one that results in moving more units into the selected unoccupied cell with the minimum cost.
5. Assign as many units as possible to this unoccupied cell.
6. Go the Step 2 and repeat the procedure until all unoccupied cells are evaluated and the value of net change, i.e., net evaluation is positive or zero.

Let us now take up an example to illustrate the Stepping Stone method.

4.4.3.1. Example. A company is spending Rs1000 on transportation of its units from three plants to four distribution centres. The availability of unit per plant and requirement of units per distribution centre, with unit cost of transportation are given as follows:

D. Centres Plants	D ₁	D ₂	D ₃	D ₄	Availability
P ₁	19	30	50	12	7
P ₂	70	30	40	60	10
P ₃	40	10	60	20	18
Requirement	5	8	7	15	

What is the maximum possible saving by optimum distribution? Use the Stepping Stone method to solve the problem.

Solution. First, we determine the initial feasible solution by applying VAM is given as follows:

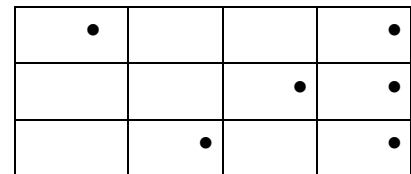
D. Centers \ Plant	D ₁	D ₂	D ₃	D ₄	Av.	Diff ₁	Diff ₂	Diff ₃	Diff ₄
P ₁	19 ⁵	30	50	12 ²	7	7	18	38	38
P ₂	70	30	40 ⁷	60 ³	10	10	10	20	20
P ₃	40	10 ⁸	6	20 ¹⁰	18	10	10	40*	-
Requirement	5*	8	7	15					
Diff ₁	21*	20	10	8					
Diff ₂	-	20*	10	8					
Diff ₃	-	-	10	8					
Diff ₄	-	-	10	48*					

In the above table, the encircled values are the allocations. The total transportation cost associated with this initial basic feasible solution is

$$= 19 \times 5 + 12 \times 2 + 40 \times 7 + 60 \times 3 + 10 \times 8 + 20 \times 10$$

$$= 95 + 24 + 280 + 180 + 80 + 200 = 859$$

Clearly, here we have 3+4-1= 6 occupied cells. These are independent as it is not possible to form any closed loop through these allocations (see Fig.).



Hence, all the conditions of applying optimality test is satisfied so we apply Stepping Stone method to obtain optimal solution.

Optimality Test using the Stepping Stone Method

We evaluate the effect of allocating one unit to each of the unoccupied cells making closed paths. Note that the unoccupied cells are (P₁,D₂), (P₁,D₃), (P₂,D₁), (P₂,D₂), (P₃,D₁) and (P₃,D₃). We have to make closed paths so that each path contains at least three occupied cells. We also have to evaluate the net change in cost for each and every unoccupied cell. Then we have to select the one unoccupied cell, which has most negative opportunity cost and allocate as many units as possible to reduce the total transportation cost. The computations are shown as follows:

Unoccupied Cell	Closed Path	Net Change in Cost (Rs)
(P ₁ , D ₂)	(P ₁ , D ₂) → (P ₁ , D ₄) → (P ₃ , D ₄) → (P ₃ , D ₂)	30-12+20-10 = 28
(P ₁ , D ₃)	(P ₁ , D ₃) → (P ₁ , D ₄) → (P ₂ , D ₄) → (P ₂ , D ₃)	50-12+60-40 = 58
(P ₂ , D ₁)	(P ₂ , D ₁) → (P ₁ , D ₁) → (P ₁ , D ₄) → (P ₂ , D ₄)	70-19+12-60 = 3
(P ₂ , D ₂)	(P ₂ , D ₂) → (P ₃ , D ₂) → (P ₃ , D ₄) → (P ₂ , D ₄)	30-10+20-60 = -20
(P ₃ , D ₁)	(P ₃ , D ₁) → (P ₁ , D ₁) → (P ₁ , D ₄) → (P ₃ , D ₄)	40-19+12-20 = 13
(P ₃ , D ₃)	(P ₃ , D ₃) → (P ₂ , D ₃) → (P ₂ , D ₄) → (P ₃ , D ₄)	60-40+60-20 = 60

The cell (P_2, D_2) has the most negative opportunity cost (net change in cost). Therefore, transportation cost can be reduced by making allocation to this unoccupied cell. This means that if one unit is shifted to this unoccupied cell through the loop shown in the fourth row of the table, then Rs20 can be saved (see Fig.4.3a). hence, we shall shift as many units as possible to the cell (P_2, D_2) through this loop. The maximum number of units that can be allocated to (P_2, D_2) through this loop is 3.

•			•
	(P_2, D_2)	•	(P_2, D_4)
	•	(P_3, D_2)	•
	•		•

(a)



(b)

This is because the shifting can be done only from the corners of the loop and more than 3 units cannot be shifted to (P_2, D_2) as explained below:

The allocations at the corners of the loops are 3, 10 and 8. If we try to shift more than 3 units, say, 4 units from the corner (P_2, D_4) , then 4 units will have to be subtracted from the corner (P_3, D_2) so that the total of the column D_4 remains unchanged. But this will give $3-4 = -1$ allocations to the cell (P_2, D_4) , which is impossible as negative allocations cannot be made.

We obtain the maximum number of units that can be allocated to the cell (P_2, D_2) through the loop as follows:

1. First, we assign (+) sign to the unoccupied cell (P_2, D_2) to be evaluated and then (-) and (+) signs alternatively to other corners of the closed loop (moving in one direction) as shown in Fig.4.3b.
2. Then we take the minimum of the values at the corners that have been assigned the negative sign. In this case, the maximum number of units that can be allocated to the cell (P_2, D_2) through the mentioned loop is the minimum of 3 and 8. It is 3. So, we write it as:

$$\min. \begin{cases} \text{the no. of units in } (P_2, D_4) = 3 \\ \text{the no. of units in } (P_3, D_2) = 8 \end{cases} = 3$$

The new table with these changes becomes:

	D ₁	D ₂	D ₃	D ₄
P ₁	19 ⁵	30	50	12 ²
P ₂	70	30 ³	40 ⁷	60
P ₃	40	10 ⁵	60	20 ¹³

In the above table, note that we have also allocated 3 units from (P₃, D₂) to (P₃, D₄) so that column D₄ remains unchanged. This leaves 5 units in the cell (P₃, D₂) and there are 13 units in (P₃, D₄). The total transportation cost associated with this solution is

$$\begin{aligned} \text{Total cost} &= 19 \times 5 + 12 \times 2 + 30 \times 3 + 40 \times 7 + 10 \times 5 + 20 \times 13 \\ &= 95 + 24 + 90 + 280 + 50 + 260 = 799 \end{aligned}$$

Now, we repeat the optimality test to see if further allocation can be made to reduce the total transportation cost. The computation for the unoccupied cells is as follows:

Unoccupied Cell	Closed Path	Net Change in Cost (Rs)
(P ₁ , D ₂)	(P ₁ , D ₂) → (P ₁ , D ₄) → (P ₃ , D ₄) → (P ₃ , D ₂)	30-12+20-10 = 28
(P ₁ , D ₃)	(P ₁ , D ₃) → (P ₁ , D ₄) → (P ₃ , D ₄) → (P ₃ , D ₂) → (P ₂ , D ₂) → (P ₂ , D ₃)	50-12+20-10+30-40 = 38
(P ₂ , D ₁)	(P ₂ , D ₁) → (P ₁ , D ₁) → (P ₁ , D ₄) → (P ₃ , D ₄) → (P ₃ , D ₂) → (P ₂ , D ₂)	70-19+12-20+10-30 = 23
(P ₂ , D ₂)	(P ₂ , D ₄) → (P ₃ , D ₄) → (P ₃ , D ₂) → (P ₂ , D ₂)	60-20+10-30 = 20
(P ₃ , D ₁)	(P ₃ , D ₁) → (P ₁ , D ₁) → (P ₁ , D ₄) → (P ₃ , D ₄)	40-19+12-20 = 13
(P ₃ , D ₃)	(P ₃ , D ₃) → (P ₂ , D ₃) → (P ₂ , D ₂) → (P ₃ , D ₂)	60-40+30-10 = 40

Since, all opportunity costs in the unoccupied cells are non-negative, the current solution is an optimal solution with total transportations cost 799. Hence the maximum saving by optimum distribution is (1000-799) = 201.

Note: The Stepping Stone method should be applied only to problems of small dimensions as it becomes quite tedious for large dimensions. To solve large dimensions problems, we use another method known as (MODI). This method may be conveniently applied to problems of small dimension as well.

4.4.4. Modified Distribution Method (MODI)

The difference between the two methods is that in the Stepping Stone method, closed loops are drawn for all unoccupied cells for determining their opportunity costs. However, in the MODI method, the opportunity costs of all the unoccupied cells are calculated and the cell with opportunity costs of all the unoccupied cells are calculated and the cell with the highest negative opportunity cost is identified without drawing any closed loop. Then only one loop is drawn for the highest negative opportunity cost. The

procedure for finding out the optimal solution of a transportation problem with the help of the MODI method is summarized in the following steps:

- (i) First find an initial feasible solution using a suitable method.
- (ii) Check optimality conditions for the current basic feasible solution if it has exactly $(m+n-1)$ independent allocations, write the cost matrix for only the allocated cells.
- (iii) Now, by using this cost matrix, determine the set of $m+n$ numbers u_i ($i=1,2,\dots,m$) and v_j ($j=1,2,\dots,n$) such that for each occupied cell (i, j) , $C_{ij} = u_i + v_j$, taking one of u_i or v_j as zero.
- (iv) Fill the vacant cells using $u_i + v_j$.
- (v) Compute all net evaluations (unit cost differences) $\Delta_{ij} = C_{ij} - (u_i + v_j)$ by subtracting the values so obtained in Step 4 from the corresponding values of the original cost matrix.
- (vi) Examine the sign of each Δ_{ij}
 - (a) If all $\Delta_{ij} > 0$, for all i, j ; then the current basic feasible solution is an optimal solution.
 - (b) If all $\Delta_{ij} \geq 0$, for all i, j and at least one Δ_{ij} is zero then the current basic feasible solution is an optimal solution and alternate optimal solution exists.
 - (c) If some of $\Delta_{ij} < 0$ for some (i, j) , the current solution is not optimal. Then select the cell having the most negative Δ_{ij} and tick it.
- (vii) Construct a closed path/loop for the unoccupied cell ticked in Step 5(ii) using the already allocated cells. At each corner of the closed path, assign plus (+) and minus (-) signs alternatively, beginning with the plus sign for the unoccupied cell. Then θ units are to be allocated to this cell and the same numbers of units are to be added and subtracted alternatively at the corners assigned plus and minus signs, respectively.

The value of θ is the maximum number of units that can be allocated to this cell through the loop. It is obtained by equating the allocations at the negative sign corners to zero. In this way, allocations have been improved.

- (viii) Write the cost matrix for only those cells which have improved allocations. Go to Step 3 and repeat the procedure until all $\Delta_{ij} \geq 0$. Calculate the associated total transportation cost.

4.4.4.1. Example. Apply the MODI method for the problem taken in example 4.4.2.1.

Solution. We find initial basic feasible solution of the problem by VAM as given in table and check optimality conditions

D. Centres \ Plant	D ₁	D ₂	D ₃	D ₄	Av.
P ₁	19 ⁵	30	50	12 ²	7
P ₂	70	30	40 ⁷	60 ³	10
P ₃	40	10 ⁸	60	20 ¹⁰	18
Requirement	5	8	7	15	35

Now we perform optimality test by applying the MODI method. First of all, we write the cost matrix for only allocated cells:

19			12	u_1
		40	60	u_2
	10		20	u_3
v_1	v_2	v_3	v_4	

Let us denote the row numbers by u_1, u_2, u_3 and column numbers by v_1, v_2, v_3 and v_4 such that

$$u_1 + v_1 = 19, \quad u_1 + v_4 = 12, \quad u_2 + v_3 = 40,$$

$$u_2 + v_4 = 60, \quad u_3 + v_2 = 10, \quad u_3 + v_4 = 20.$$

Taking $u_1=0$, we have $v_1 = 19$ and $v_4 = 12$ from the first two equations.

Putting the value of v_4 in the fourth equation, we have $u_2 = 48$.

Similarly, we find $u_3 = 8, v_2 = 2$ and $v_3 = -8$

Using these values, we fill the vacant cells of the above table using $C_{ij} = u_i + v_j$ and put dots in the already filled cells so that these cells are not considered again.

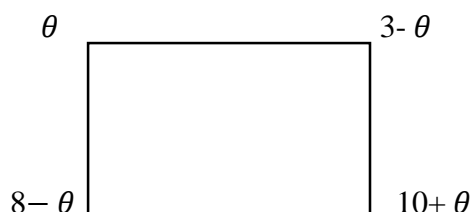
	$v_1 = 19$	$v_2 = 2$	$v_3 = -8$	$v_4 = 12$
$u_1 = 0$	•	2	-8	•
$u_2 = 48$	67	50	•	•
$u_3 = 8$	27	•	0	•

Now, subtracting these values from the corresponding values of the original cost matrix, we have the net evaluations, i.e., $\Delta_{ij} = C_{ij} - (u_i + v_j)$

•	$30-2=28$	$50-(-8)=58$	•
$70-67=3$	$30-50=-20$	•	•
$40-27=13$	•	$60-0=60$	•

Since Note that the cell (P_2, D_2) has the most negative opportunity cost (net change in cost).

Therefore, the transportation cost can be reduced by making allocation to this unoccupied cell. This means that if one unit is shifted to this unoccupied cell through the closed loop formed, beginning from this cell and using allocated cells, Rs20 can be saved. We form the loop beginning from this cell, i.e., the cell (P_2, D_2) (see Fig. 4.4). We shift θ units to this unoccupied cell through the loop and add and subtract θ from the cells at the other corners of the loop which are assigned '+' and '-' signs. So, we get



The maximum number of the units that can be allocated to the cell (P_2, D_2) through this loop is given by the minimum of the solution of the equation $3 - \theta = 0$ and $8 - \theta = 0$, i.e.,

$$\theta = \min. \begin{cases} \text{the no. of units in } (P_2, D_4) = 3 \\ \text{the no. of units in } (P_2, D_2) = 8 \end{cases} = 3$$

So, with the improved allocations, the table now becomes:

19 ₅	30	50	12 ₂
70	30 ₃	40 ₇	60
40	10 ₅	60	20 ₁₃

Thus, the total cost of transportation for this set

$$\begin{aligned} &= 19 \times 5 + 12 \times 2 + 30 \times 3 + 40 \times 7 + 10 \times 5 + 20 \times 13 \\ &= 95 + 24 + 90 + 280 + 50 + 260 = 799 \end{aligned}$$

Now, let us apply the optimality test to the improved solution. Proceeding in the same way as in the first iteration, first of all, we write the cost matrix for only allocated cells:

19			12	u_1
	30	40		u_2
	10		20	u_3
v_1	v_2	v_3	v_4	

Let us denote the row numbers by u_1, u_2, u_3 and column numbers by v_1, v_2, v_3 and v_4 such that

$$\begin{aligned} u_1 + v_1 &= 19, & u_1 + v_4 &= 12, & u_2 + v_3 &= 40, \\ u_2 + v_2 &= 30, & u_3 + v_2 &= 10, & u_3 + v_4 &= 20. \end{aligned}$$

Taking $u_1 = 0$, we have $v_1 = 19$ and $v_4 = 12$ from the first two equations.

Putting the value of v_4 in the fourth equation, we have $u_3 = 8$.

Similarly, $u_2 = 8$, $v_2 = 2$ and $v_3 = 12$

Using these values, we fill all the vacant (unoccupied) cells of the table using $C_{ij} = u_i + v_j$ for each unoccupied cell and put dots in the already filled cells so that these cells are not considered again.

	$v_1 = 19$	$v_2 = 2$	$v_3 = 12$	$v_4 = 12$
$u_1 = 0$	•	2	-8	•
$u_2 = 28$	47	•	•	40
$u_3 = 8$	27	•	20	•

Now, subtracting these values from the corresponding values of the original cost matrix, we have the net evaluations, i.e., $\Delta_{ij} = C_{ij} - (u_i + v_j)$

•	28	38	•
23	•	•	20
13	•	40	•

Since none of the net evaluation is negative, this solution is optimal. Thus, the total minimum transportation cost is Rs799 and the maximum saving = (1000-799) = 201.

4.5. UNBALANCED TRANSPORTATION PROBLEM

The transportation problem wherein the total capacity of all sources and total requirement (demand) of all destinations are not equal is called the unbalanced transportation problem. An unbalanced transportation problem may occur in two different forms (i) Excess of availabilities (ii) Shortage in availabilities. In case (i), we add a dummy destination in the transportation table with zero transportation cost. The requirement of this dummy destination is assumed to be equal to the difference of total availability of sources and total demand so that the problem becomes balanced. Similarly, in case (ii) we introduce a dummy source and take the corresponding steps. The following example explain the whole procedure.

4.5.1. Example. A company has factories at A, B and C which supply warehouses at D, E, F, and G. The monthly factory capacities are 160, 150 and 190 units, respectively. Monthly warehouse requirements are 80, 90, 110 and 160, respectively. Unit shopping costs (in rupees) are as follows:

From	To				
		D	E	F	G
	A	42	48	38	37
	B	40	49	52	51
	C	39	38	40	43

Determine the optimum distribution for this company to minimize shipping costs.

Solution. Here the total capacity of sources (factories) is $160 + 150 + 190 = 500$. It is greater than the total requirement of all the destinations (warehouses) which is $80 + 90 + 110 + 160 = 440$.

Therefore, we added dummy destination in the transportation table with zero transportation cost and take 60 as its requirement. Thus, the problem becomes balanced that is the total capacity and total requirement are equal the balanced., the problem is as follows:

From	To						Capacity
		D	E	F	G	H	
	A	42	48	38	37	0	160
	B	40	49	52	51	0	150
	C	39	38	40	43	0	190
Requirement		80	90	110	160	60	500

We can solve this problem using VAM to determine the basic feasible solution and then use MODI method to find the optimal solution. We have solved this problem in Example 3. You may like to try the remaining solution yourself.

The optimum allocation is:

160 to (A, F), 80 to (B, D), 10 to (B, E), 60 to (B, G), 80 to (C, E) and 110 to (C, F).

Note: If the number of allocations in the basic feasible solution is less than $m + n - 1$, we first go through the special case of degeneracy and then apply the test of optimality.

4.6. DEGENERACY

Degeneracy occurs in the transportation problem if the number of occupied cells is less than $m + n - 1$. To resolve degeneracy, we introduce an infinitesimally allocation e (say) in the least cost and independent cell so that number of occupied cells becomes $m+n-1$. Also notice that if 'e' is added to or subtracted from any quantity, the quantity remains unaltered.

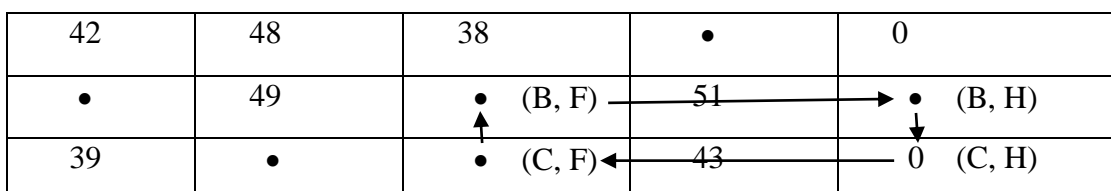
Remark: We do not write 0 in place of e as it is one of the allocations and will not come into counting if it is written as 0 so we write e and not 0 in its place and count it as one of the allocations.

Degeneracy occurs in Example 4.5.1. To solve this problem, we proceed as follows:

Using Vogel Approximation method, we get the basic feasible solution as:

Warehouse Factory	D	E	F	G	H	Av.	D ₁	D ₂	D ₃	D ₄
A	42	48	38	37 ⁽¹⁶⁰⁾	0 ^(e)	160	37	1	1	1
B	40 ⁽⁸⁰⁾	49	52 ⁽¹⁰⁾	51 ⁽⁶⁰⁾	0 ⁽⁶⁰⁾	150	40*	9	11*	1
C	39	38 ⁽⁹⁰⁾	40 ⁽¹⁰⁰⁾	43	0	190	38	1	3	3
Requirement	80	90	110	160	60	500				
Diff ₁	1	10	2	6	0					
Diff ₂	1	10*	2	6	-					
Diff ₃	1	-	2	6	-					
Diff ₄	-	-	2	6*	-					

Here the number of allocations is 6 but the number of allocations should be $3 + 5 - 1 = 7$ to perform optimality test. Hence it is the case of degeneracy to resolve such a problem, we introduce infinitesimally small allocation e in the least cost and independent cell, i.e., the cell (A,H). Another cell (C, H) also has same least cost as the cell (A, H). But this cell is not independent because a loop (C,H) → (C,F) → (B,F) → (B,H) can be formed beginning with the cell as shown in figure 4.5. Now the number of allocations is 7, i.e., as many as required for performing the optimality test.



Thus, applying optimality test by MODI method, the improved allocations after first iteration will be:

42	48	38 _e	37 ₁₆₀	0
40 ₈₀	49	52 ₁₀	51	0 ₆₀
39	38 ₉₀	40 ₁₀₀	43	0

Again, applying the optimality test, the improved solution is

	D	E	F	G	H
A	42	48	38 _e	37 ₁₆₀	0
B	40 ₈₀	49 ₁₀	52	51	0 ₆₀
C	39	38 ₈₀	40 ₁₁₀	43	0

Again, applying the optimality test, you will see that these allocations give the optimal solution and the total transportation cost = rupees 17050.

4.7. CHECK YOUR PROGRESS

1. Explain transportation problem. Is this a linear programming problem?
2. Explain method for obtaining initial basic feasible solution of Transportation problem:
 - (i) The North-West Corner Rule,
 - (ii) Vogel's Approximation Method (VAM),
 - (iii) The least Cost Method.
3. Describe the computational procedure of the following optimality tests for solving transportation problem:
 - (i) Stepping stone method,
 - (ii) MODI method.
4. Write a short note on unbalanced transportation problem.
5. What is degeneracy in a transportation problem?

4.8. SUMMARY

In this chapter, we have introduced transportation problems. As one can express these problems in terms of LPPs so transportation problems is also considered as one of the sub – classes of Linear Programming Problems and the objective of the problem is to determine the optimal amount to ship/ transport from each origin to destination. Here, we have explained how to formulate mathematically a transportation problem, finding initial basic feasible solution, performing optimality test and moving towards optimal solution.

In this chapter, we have discussed three methods for finding initial basic feasible solution namely North – West Corner Rule, Least Cost Method and Vogel's Approximation Method. Vogel's approximations method is preferred over other methods of finding the initial basic feasible solutions as it generally gives the solution closest to the optimum solution. After finding the initial basic feasible solution, optimality test is performed by applying the Stepping Stone Method or Modified Distribution Method (MODI) to find whether the obtained feasible solution is optimal or not. In the last we have discussed the case of unbalanced transportation problems and degeneracy.

5

ASSIGNMENT PROBLEMS

Structure

- 5.1. Introduction
- 5.2. Assignment Problems
- 5.3. Hungarian Method
- 5.4. Unbalanced Assignment Problem
- 5.5. Case of Maximization of an Assignment Problem
- 5.6. Travelling Salesman Problem
- 5.7. Check Your Progress
- 5.8. Summary

5.1. INTRODUCTION

Assignment problem is a special type/case of transportation problem and hence is of linear programming problem which deals with the allocation of the various resources to the various activities on one to one basis. It does it in such a manner that the cost or time involved in the process is minimum and profit or sale is maximum. In chapter 4, we have already discussed methods of finding optimal solution for the given problem but here we discuss another method namely Hungarian method for solving an assignment problem. Hungarian method is shorter and easier than stepping stone and MODI methods which we have discussed in previous chapter. In this chapter, we shall explain the assignment problems including travelling salesman problem and apply Hungarian method for solving these problems.

5.1.1. Objectives

After studying this chapter, reader should be able to explain the following concepts like:

- Assignment Problems
- Hungarian Method.
- Unbalanced Assignment Problem
- Case of Maximization of an Assignment Problem
- Travelling Salesman Problem

5.2. ASSIGNMENT PROBLEMS

An assignment problem may be considered as a special type of transportation problem in which there are as many jobs/sources as the number of machines/destinations so that the jobs can be assigned to machines in a one-to-one way only. The capacity of each source as well as the requirement of each destination is taken as 1. The main difference between an assignment problem and transportation problem is that in the case of assignment problem, the given matrix must necessarily be a square matrix which is not the condition for a transportation problem.

Let there be n persons and n jobs and let C_{ij} represents the amount of time taken by i^{th} person to complete the j^{th} job then our objective is assignment of jobs on one-to-one basis in such a way that the total cost is minimum. The assignment problem can be stated in the form of an $n \times n$ matrix of real numbers called the cost matrix as given in the following table:

Person	Job					
	1	2	...	j	...	n
1	C_{11}	C_{12}	...	C_{1j}	...	C_{1n}
2	C_{21}	C_{22}	...	C_{2j}	...	C_{2n}
.
.
i	C_{i1}	C_{i2}	...	C_{ij}	...	C_{in}
.
.
n	C_{n1}	C_{n2}	...	C_{nj}	...	C_{nn}

Let x_{ij} denote the j^{th} job assigned to the i^{th} person. Then, mathematically, the assignment problem can be stated as follows:

$$\text{Minimize } Z = \sum_{i=1}^n \sum_{j=1}^n c_{ij}x_{ij}, \quad \text{where } i=1, 2, \dots, n \text{ and } j=1, 2, \dots, n.$$

subject to

$$x_{ij} = \begin{cases} 1 & \text{if the } i^{th} \text{ person is assigned the } j^{th} \text{ job} \\ 0 & \text{if the } i^{th} \text{ person is not assigned the } j^{th} \text{ job} \end{cases}$$

Since one job assigned to one person only, we have

$$x_{i1} + x_{i2} + \dots + x_{in} = 1, i = 1, 2, \dots, n$$

$$x_{1j} + x_{2j} + \dots + x_{nj} = 1, j = 1, 2, \dots, n$$

Note: The constant c_{ij} in the above problem represents time, in many situations it may be cost or some other parameter which is to be minimized in the assignment problem under consideration.

Remark: An assignment problem can be looked upon as a special type of transportation problem in which the jobs stands for sources, the machines for destinations and all the availabilities and requirements are equal to one.

5.3. HUNGARIAN METHOD

Hungarian method is also known as Reduced Matrix Method, it is an efficient method for solving assignment problems. Hungarian method is developed by Hungarian mathematician D. Konig. The step by step procedure for obtaining an optimal solution of an assignment problem are as follows:

1. Develop the cost table from the given problem then check whether the given matrix is square i.e. is number of sources/machines is equal to the number of destinations/jobs. If not, make it square by adding a suitable number of dummy row (or column) with 0 cost/time element.
2. Locate the smallest cost element in each row of the given cost matrix and then subtract the smallest element of each column from every element of that column.
3. In the resulting cost matrix, locate the smallest element in each column and subtract the smallest element of each column from every element of that column.
4. In the modified matrix, search for an optimal assignment as follows:
 - a) Examine the row successively until a row with exactly one zero is found. Draw a rectangle around this zero like 0 and cross out all other zeroes in the corresponding column. Proceed in this way until all the row have been considered. If there is more than one zero in any row, don't touch that row, pass on to the next row.
 - b) Repeat step (a) above for the columns of the resulting cost matrix.
 - c) If a row or column of the reduced matrix contains more than one zeroes, arbitrarily choose a row or column having the minimum number of zeroes. Arbitrarily select any zero in the row or column so chosen. Draw a rectangle around it and cross out all the zeroes in the corresponding row and column. Repeat steps (a), (b), and (c) until all the zeroes have either been assigned (by drawing a rectangle around them) or crossed.
 - d) If each row or column of the resulting matrix has one and only one assigned zero, i.e. number of assigned zeroes are equal to the number of rows/columns, then the optimum assignment is made in the cells corresponding to 0 and the optimum solution of the problem is attained and we can stop here. Otherwise, go to the next step.
5. Draw the minimum number of horizontal and/or vertical lines through all the zeroes as follows:
 - i. Mark (\surd) the rows in which assignment has not been made.
 - ii. Mark (\surd) column, that have zeroes in the marked rows.

- iii. Mark (\surd) rows(not already marked) which have assignments in marked columns. Then mark (\surd) columns, which have zeroes in newly marked rows, if any. Mark (\surd) rows (not already marked), which have assignments in these newly marked columns.
6. Revise the cost matrix as follows:
- i. Find the smallest elements not covered by any of the lines.
 - ii. Subtract this from all the uncovered elements and add it to the elements at the intersection of the two lines.
 - iii. Other elements covered by the lines remain unchanged.
7. Repeat the procedure until an optimal solution attained.

Note: By drawing lines through all the unmarked rows and marked columns, we will get the required minimum number of lines.

The following example illustrate the method.

5.3.1. Example. A computer centre has four expert programmers & needs to develop four application programmes. The head of the computer Centre, estimates the computer time (in minutes) required by the respective experts to develop the application programmes as follows:

PROGRAMMES PROGRAMMERS	A	B	C	D
1	120	100	80	90
2	80	90	110	70
3	110	140	120	100
4	90	90	80	90

Find the assignment pattern that minimize the time required to develop the application programmes.

Solution. Let us subtract the minimum element of each row from every element of that row. Note that the minimum element in the first row is 80. So, 80 is subtracted from every element of the first row. Similarly, we obtain the elements of the other rows of the resulting matrix. Thus, the modified matrix is:

	A	B	C	D
1	40	20	0	10
2	10	20	40	0
3	10	40	20	0
4	10	10	0	10

Let us now subtract the minimum element of each column from every element of that column in the resulting matrix. The minimum element in the first column is 10. So, 10 is to be subtracted from every element of the first column. Similarly, we obtain the elements of the other columns of the resulting matrix. Thus, the resulting matrix is:

	A	B	C	D
1	30	10	0	10
2	0	10	40	0
3	0	30	20	0
4	0	0	0	10

Now starting from the first row onward, we draw a rectangle around the zero in each row having a single zero and cross all other zeros in the corresponding column. Here, in the very first row we find a single zero. So, we draw a rectangle around it and cross all the other zeroes in the corresponding column. We get

	A	B	C	D
1	30	10	0	10
2	0	10	40	0
3	0	30	20	0
4	0	0	0	10

In the second, third and fourth row, there is no single zero. Hence, we move column – wise. In the second column, we have a single zero. Hence, we draw a rectangle around it and cross all other zeroes in the corresponding row. We get

	A	B	C	D
1	30	10	0	10
2	0	10	40	0
3	0	30	20	0
4	0	0	0	10

In the matrix above, there is no row or column, which has a single zero. Therefore, we first move row – wise to locate the row having more than one zero. The second row has two zeroes. So, we draw a rectangle arbitrarily around one of these zeroes and cross the other one. Let us draw a rectangle around the zero in the cell (2, A) and cross the zero in cell (2, D). We cross out the other zeroes in the first column. Note that we could just as well have selected zero in the cell (2, D), drawn a rectangle around it and crossed all other zeroes. This would have led to an alternative solution.

In this way, we are left with only one zero in every row and column around which a rectangle has been drawn. This means that we have assigned only one operation to one operator. Thus, we get the optimum solution as follows:

	A	B	C	D
1	30	10	0	10
2	0	10	40	0
3	0	30	20	0
4	0	0	0	10

Note that the assignment of jobs should be made on the basis of the cells corresponding to the zeroes around which rectangles have been drawn. Therefore, the optimum solution for this problem is:

$$1 \rightarrow C, 2 \rightarrow A, 3 \rightarrow D, 4 \rightarrow B$$

This means that programmer 1 is assigned programme C, programmer 2 is assigned programme A, and so on. The minimum time taken in developing the programmes is

$$= 80 + 80 + 100 + 90 = 350 \text{ min.}$$

5.3.2. Example. A company is producing a single product and selling it through five agencies situated in the different cities. All of a sudden, there is a demand for the product in five more cities that do not have any agency of the company. The company is faced with the problem of deciding on how to assign the existing agencies to dispatch the product to the additional cities in such a way that the travelling distance is minimized. The distances (in km) between the surplus and the deficit cities are given in the following distance matrix:

Deficit city Surplus city	I	II	III	IV	V
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

Determine the optimum assignment schedule.

Solution: Subtracting the minimum element of each row from every element of that row and then subtracting the minimum element of each column from every element of that column, we have

	I	II	III	IV	V
A	30	0	35	30	15
B	15	0	0	10	0
C	30	0	35	30	20
D	0	0	20	0	5
E	20	0	25	15	15

We now assign zeroes by drawing rectangles around them as explained in above example. Thus, we get

	I	II	III	IV	V
--	----------	-----------	------------	-----------	----------

A	30	0	35	30	15
B	15	0	0	10	0
C	30	0	35	30	20
D	0	0	20	0	5
E	20	0	25	15	15

Since the number of assignments is less than number of rows (or columns), we proceed from step 5 onwards of the Hungarian method described as follows:

- i. We mark (\checkmark) the rows in which the assignment has not been made . These are the 3rd & 5th row.
- ii. We mark (\checkmark) the columns which have zeroes in the marked rows . This is the 2nd column.
- iii. We mark (\checkmark) the rows which have assignments in marked columns . This is the 1st row.
- iv. Again we mark (\checkmark) the columns which have zeroes in the newly marked row . This is the 2nd column (which has been already marked).

There is no other such column. So, we have

	I	II	III	IV	V	
A	30	0	35	30	15	\checkmark
B	15	0	0	10	0	
C	30	0	35	30	20	\checkmark
D	0	0	20	0	5	
E	20	0	25	15	15	\checkmark
		\checkmark				

We draw straight lines through unmarked rows and marked columns as follows:

	I	II	III	IV	V	
A	30	0	35	30	15	\checkmark
B	15	0	0	10	0	—
C	30	0	35	30	20	\checkmark
D	0	0	20	0	5	—
E	20	0	25	15	15	\checkmark
		\checkmark				

We proceed as follows, as explained in the step 6 of the Hungarian method:

- 1) We find the smallest element in the matrix not covered by any of the lines. It is 15 in this case.
- 2) We subtract the number '15' from all uncovered elements and add it to the elements at the intersection of the two lines.

3) Other elements covered by the lines remain unchanged. Thus, we have

	I	II	III	IV	V
A	15	0	20	15	0
B	15	15	0	10	0
C	15	0	20	15	5
D	0	15	20	0	5
E	5	0	10	0	0

We repeat steps 1 to 4 of the Hungarian method and obtain the following matrix:

	I	II	III	IV	V
A	15	∞	20	15	0
B	15	15	0	10	∞
C	15	0	20	15	5
D	0	15	20	∞	5
E	5	∞	10	0	∞

Since each row and each column of this matrix has one and only one assigned 0, we obtain the optimum assignment schedule as follows:

A → V, B → III, C → II, D → I, E → IV

Thus, the minimum distance is $200+130+110+50+80 = 570$ km.

5.3.3. Exercise. A solicitor’s firm employs typists on an hourly piece – rate basis for their daily work. There are five typists for service and their charges and speeds are different. According to the contract, only one job is given to one typist. Find the least cost allocation for the following data:

	P	Q	R	S	T
A	85	75	65	85	75
B	90	180	66	90	78
C	75	66	57	75	69
D	80	72	60	80	72
E	76	64	56	72	68

Answer. Applying the Hungarian method of solving an assignment problem, we finally get

	P	Q	R	S	T
A	2	4	2	4	0
B	4	106	0	6	∞
C	∞	0	2	2	2
D	0	4	∞	2	∞
E	2	2	2	0	2

Thus, the least cost allocation is given by:

$$A \rightarrow T, B \rightarrow R, C \rightarrow Q, D \rightarrow P, E \rightarrow S$$

and the total minimum cost is Rs. $(75+66+66+80+72)$

=Rs.359.

5.4. UNBALANCED ASSIGNMENT PROBLEM

Some assignment problems may be unbalanced, i.e. the number of machines may be different from the number of jobs. In this case, in the obtained matrix the number of rows is not equal to the number of columns and the problem said to be an unbalanced Assignment problem. Such a problem is handled by introducing dummy row(s) if the number of rows is less than the number of columns and dummy column(s) if the number of columns is less than the number of rows. All the elements of such a dummy row/column are taken as zero. After creating dummy rows or columns, we get a balanced assignment problem and now we solved it by Hungarian method.

The following example will make the procedure clear.

5.4.1. Example. To stimulate interest and provide an atmosphere for intellectual discussion, the faculty of mathematical sciences in an institute decides to hold special seminars four contemporary topics - Statistics, Operations Research, Discrete Mathematics, Matrices. Each such seminar is to be held once a week. However, scheduling these seminars (one for each topic and not more than one seminar per a day) has to be done carefully so that the numbers of students unable to attend is kept to a minimum. A careful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows:

	Statistics	Operations Research	Discrete mathematics	Matrices
Monday	50	40	60	20
Tuesday	40	30	40	30
Wednesday	60	20	30	20
Thursday	30	30	20	30
Friday	10	20	10	30

Find an optimal schedule for the seminars. Also find the number of students who will be missing at least one seminar.

Solution. Here the number of rows is 5 and the number of columns is 4. Therefore, the given assignment problem is unbalanced. As the number of columns is one less than the number of rows, we introduce one dummy column to convert the given assignment problem into a balanced problem. The number of students in each cell of this column is taken as zero. Thus, the problem takes the following form:

	Statistics	Operations Research	Discrete mathematics	Matrices	Dummy
Monday	50	40	60	20	0
Tuesday	40	30	40	30	0
Wednesday	60	20	30	20	0
Thursday	30	30	20	30	0
Friday	10	20	10	30	0

Now, on applying the Hungarian method (Steps 1 to 4), we get

	Statistics	Operations Research	Discrete mathematics	Matrices	Dummy
Monday	40	20	50	0	∞
Tuesday	30	10	30	10	0
Wednesday	50	0	20	∞	∞
Thursday	20	10	10	10	∞
Friday	0	∞	∞	10	∞

Since the number of assigned zeroes < number of rows, we apply Step 5 of the Hungarian method and draw the minimum number of horizontal/ vertical lines that cover all the zeros as shown in the following table:

	Statistics	Operations Research	Discrete mathematics	Matrices	Dummy
Monday	40	20	50	0	∞
Tuesday	30	10	30	10	0
Wednesday	50	0	20	∞	∞
Thursday	20	10	10	10	∞
Friday	0	∞	∞	10	∞

We select the minimum element from amongst the uncovered elements, which in 10 in this case. We subtract this element, i.e., 10 from each uncovered element and add it to the elements which lie at the intersection of the horizontal/vertical lines. Other covered elements will remain unaltered. Thus, the resulting matrix is:

40	20	50	0	10
20	0	20	0	0
50	0	20	0	10
10	0	0	0	0
0	0	0	10	10

Now on applying the Hungarian method, we have

40	20	50	0	10
20	∞	20	∞	0
50	0	20	∞	10
10	∞	0	∞	∞
0	∞	∞	10	10

Since each row and each column of the matrix has one and only one assigned 0, optimum assignment is made in the cells containing those zeroes around which rectangles have been drawn as Monday → Matrices, Wednesday → Operations Research, Thursday → Discrete Mathematics, Friday → Statistics. The total number of students who will be missing at least one seminar = $20 + 20 + 20 + 10 = 70$

5.5. CASE OF MAXIMIZATION OF AN ASSIGNMENT PROBLEM

All problems dealt with so far were all cost-minimizing problems but assignment problem exists with profit maximization problem also. For example, profits (or anything else like revenues), which need maximization may be given in the cells instead of costs/ times. The method of solving such problems is a simple modification of the method of solving cost-minimizing assignment problems. To solve such a problem, we find the opportunity loss matrix by subtracting the value of each cell from the largest value chosen from amongst all the given cells. When the value of a cell is subtracted from the highest value, it gives the loss of amount caused by not getting the opportunity which would have given the highest value. The matrix so obtained is handled in the same way as the minimization problem. The following example illustrate the method.

5.5.1. Example. Five salesmen are to be assigned to five districts. Estimates of sales revenue (in thousands) for each salesman are given as following:

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	27
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment pattern that maximizes the sales revenue.

Solution. Since we are to maximize the sales revenue, we need to convert it into minimization form before applying the Hungarian method. For this, we obtain the opportunity loss matrix by subtracting every element in the given table from the largest element is 41. Thus, we obtain the following opportunity loss matrix:

9	3	1	13	1
1	17	13	20	5
0	14	8	11	4
19	3	0	5	5
12	8	1	6	2

Now, we apply the Hungarian method (Steps 1 to 4) and finally obtain the following result matrix:

	A	B	C	D	E
1	8	0	∞	7	∞
2	0	14	12	14	4
3	∞	12	8	6	4
4	19	1	0	∞	5
5	11	5	∞	0	1

Since the number of assigned zero is less than the number of rows, we apply Step 5 of the Hungarian method and draw the minimum number of horizontal/vertical lines that cover all the zeroes as shown in the following table:

	A	B	C	D	E	
1	∞	0	∞	7	∞	
2	0	14	12	14	4	√
3	∞	12	8	6	4	√
4	19	1	0	∞	5	
5	11	5	∞	0	1	
	√					

Let us now, select the minimum element from amongst the uncovered elements, which is 4 in the case. We subtract the element 4 from each of the uncovered elements and add it to the elements which lie at the intersection of the horizontal and vertical lines. Other covered elements remain unaltered. Then applying the Hungarian method to the resulting matrix. We get

	A	B	C	D	E
1	12	0	∞	7	∞
2	0	10	8	10	∞
3	∞	8	4	2	0
4	23	1	0	∞	5
5	15	5	∞	0	1

Since the number of assigned zeroes is equal to the number of rows, the optimum assignment has been attained and is given as

$1 \rightarrow B, 2 \rightarrow A, 3 \rightarrow E, 4 \rightarrow C, 5 \rightarrow D$

Thus, the maximum sales revenue = $38 + 40 + 37 + 41 + 35$ thousand rupees
= 191 thousand rupees.

5.6. TRAVELLING SALESMAN PROBLEM

Consider a travelling salesman who has to visit a certain number of cities allotted to him. He wishes to visit each city once and only once and arrive back in the city from where he started. He knows the distances (or cost or time) of journey between every pair of cities, and he wishes to determine the tour schedule that represents the least distance/cost/time. Such type of problems can be solved by the assignment algorithm.

The difference between a travelling salesman and assignment problem is that in an assignment problem, different destinations are assigned to different sources but in a travelling salesman problem, a destination is assigned to a source. Then this destination becomes another source to which we assign another destination, which in turn becomes another source, and so on. Let us explain this point further with the help of an example.

5.6.1. Example: A travelling salesman has to visit five cities. He wishes to start from a particular city, visit each city once and then return to his starting point. The travelling times for each city from a particular city is given below

To \ From	A	B	C	D	E
A	∞	5	8	4	5
B	5	∞	7	4	5
C	8	7	∞	8	6
D	4	4	8	∞	8
E	5	5	6	8	∞

What is the sequence of visits of the salesman so that the total travelling time is minimized?

Solution: Applying the Hungarian method to this problem, we get

	A	B	C	D	E
A	∞	5	2	0	5
B	0	∞	1	4	5
C	2	1	∞	3	0
D	4	0	3	∞	4
E	5	5	0	4	∞

As per the above assignment, the salesman should travel from A to D, D to B, B to A i.e., $A \rightarrow D \rightarrow B \rightarrow A$.

The above solution is not a complete solution of the travelling salesman problem as the salesman returns to A without travelling through all the cities. so we proceed as follows.

Since the assignment of zeroes has not given the solution of the travelling salesman problem, we bring the next minimum non-zero element into the solution. Thus, we obtain the next best solution by bringing 1

into the solution. had there not been 1 in any cell, we would have taken the minimum, but greater than 1 value from amongst all the values of the tables. here we have 1 and it appears in two places in this problem. one of these is chosen arbitrarily. let us choose the cell (B, C) and from a rectangle around the values in this cell and cross out the zeroes in its row and column. now we apply the Hungarian method for the assignment of zeroes. Thus, we have

	A	B	C	D	E
A	∞	5	2	0	6
B	3	∞	1	2	3
C	2	1	∞	3	0
D	0	2	3	∞	4
E	3	0	4	4	∞

Alternatively, we get

	A	B	C	D	E
A	∞	5	2	2	0
B	3	∞	1	0	3
C	2	1	∞	3	4
D	0	2	3	∞	4
E	3	4	0	4	∞

In the case of the first alternative, the optimum assignment is A→D→A, but this is not the solution of the travelling salesman problem.

In the case of the second alternative, the optimum assignment is

$$A \rightarrow E \rightarrow C \rightarrow B \rightarrow D \rightarrow A$$

This is the complete solution for the problem as starting from A the salesman visits all the other cities and returns to A. The minimum time taken by him to travel to all the cities is 5+6+7+4+4=26hrs.

5.6.2. Exercise. Solve the following travelling salesman problem so as to minimize the cost per cycle:

To \ From	A	B	C	D	E
A	∞	3	6	2	3
B	3	∞	5	2	3
C	6	5	∞	6	4
D	2	2	6	∞	6
E	3	3	4	6	∞

What is the sequence of visits of the salesman so that travelling time is minimized?

Answer. Applying the Hungarian method of solving an assignment problem, we reduce the cost matrix and make assignments in rows and columns having single zeroes.

	A	B	C	D	E	
A	∞	\times	3	0	1	\checkmark
B	1	∞	2	\times	1	\checkmark
C	2	1	∞	2	0	
D	\times	0	3	∞	4	
E	\times	\times	0	3	∞	
				\checkmark		

Now, we draw the minimum number of lines to cover all the zeroes. Then we subtract the lowest element from all the elements not covered by any of lines and add the same at the intersection of two lines. We have

	A	B	C	D	E	
A	∞	\times	2	0	\times	\checkmark
B	0	∞	1	\times	\times	\checkmark
C	2	1	∞	3	0	
D	\times	0	3	∞	4	
E	\times	\times	0	4	∞	
				\checkmark		

As per the above assignment, the salesman should travel from A to D, D to B, B to A,

i.e., $A \rightarrow D \rightarrow B \rightarrow A$.

The above solution is not a complete solution of the travelling salesman problem as the salesman returns to A without travelling through all the cities. So, we proceed as follows:

Since the assignment of zeroes has not given the solution of the travelling salesman problem, we bring the next minimum non-zero element in the solution.

Thus, we obtain the next best solution by bringing 1 into the solution. Had there not been 1 in any cell, we could have taken the minimum, but greater than 1 value from amongst all the values of the table. Here, we have 1 and it appears in two places in this problem. one of these is chosen arbitrarily. Let us choose the cell (B, C) and form a rectangle around the value in this cell and cross out the zeroes in its row and column. Now we apply the Hungarian method for the assignment of zeroes. Thus, we have

	A	B	C	D	E
A	∞	\times	2	0	\times
B	\times	∞	1	\times	\times
C	2	1	∞	3	0
D	0	\times	3	∞	4
E	\times	0	\times	4	∞

Alternatively, we get

	A	B	C	D	E
A	∞	∞	2	∞	0
B	∞	∞	1	0	∞
C	2	1	∞	3	∞
D	0	∞	3	∞	4
E	∞	∞	0	4	∞

In this case of the first alternative, the optimum assignment is $A \rightarrow D \rightarrow A$, but this is not the solution of the travelling salesman problem.

In this case of the second alternative, the optimum assignment is

$$A \rightarrow E \rightarrow C \rightarrow B \rightarrow D \rightarrow A, \text{ i.e., } 3 + 4 + 5 + 2 + 2 = 16 \text{ hr}$$

5.7. CHECK YOUR PROGRESS

1. Define an assignment problem.
2. Explain the difference between a transportation problem and assignment problems.
3. Explain step by step procedure of Hungarian method to solve the assignment problem.
4. What is an unbalanced assignment problem?
5. Explain briefly travelling salesman problem.

5.8. SUMMARY

In this chapter, we have observed that an assignment problem may be considered as a special type of transportation problem in which the capacity of each of the sources as well as the requirements of each of the destination is taken as 1. In an assignment problem, the given matrix must necessarily be a square matrix, which is not the condition for a transportation problem. Such problem is solved as Hungarian method, which is shorter and easy compared to any other method of finding the optimal solution. We have discussed the concept of unbalanced transportation problem; such problem is handled by introducing dummy rows/columns. In last, we have introduced travelling salesman problem.

6

QUEUEING THEORY

Structure

- 6.1. Introduction.
- 6.2. Basic Concepts of Queueing Theory.
- 6.3. Fundamental Structure of a Queueing System.
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- 6.5. M/M/1 or M/M/1: ∞ /FCFS Queueing Model.
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6.1. INTRODUCTION

Queueing or waiting in a line is a common situation occurring in everyday life. We wait in queues in ticket booths, bus stops, post offices, banks, traffic lights and so on. We can think of many more situations. In general, a queue is formed when there are customers who require some sort of services and the queueing problem is identified by the presence of a group of customers who arrive randomly to receive some service.

The customer may be a person, machine, vehicle or anything else which requires service. The objective of a queueing model is to find out the optimum service rate and the number of servers so that the average cost of being in queueing system and the cost of service are minimized.

The objective of a queueing system is to find ways of reducing the time spent in waiting by the customer and at the same time optimizing the cost to the service provider. Sometimes Queueing Theory, also

reference as the waiting line theory. It was developed in 1909 when A.K. Erlang made an effort to analyse telephone traffic congestion. It can be applied to wide variety of operational situations whenever customer's expectations do not match with objectives of servers due to one's inability to predict accurately the arrival and service time of customers. The purpose of queueing analysis is to provide information to evaluate an acceptable level of service and service capacity since providing too much service capacity is costly (because of employees or equipment). In fact, providing too little service capacity is also costly.

In this Chapter, we discuss the concepts of stochastic processes, Poisson process and birth-death process. We explain the basic component, fundamental structure and operating characteristics of a queueing system. Also, we describe the M/M/1, queueing models.

6.1.1. Objectives. The objective of these contents is to provide some important concepts/results to the reader like:

- Poisson and birth-death process.
- Fundamental structure and operating characteristics of a queueing system.
- M/M/1 Queueing Model.

6.2. BASIC CONCEPTS OF QUEUEING THEORY

For studying queueing systems, one should be familiar with the probability theory, specially the concept of random variable and its probability distribution such as Poisson distribution, Exponential distribution. Let us have a glance on some basic definitions.

Consider an experiment whose outcome is not uniquely determined. In such a situation an observed outcome of the experiment is one from a set of possible outcomes.

Sample point: An outcome of an experiment is called a sample point.

Sample Space: The set of all possible outcomes of a random experiment is called sample space.

Events: Subsets of the sample space are called events.

Random Variable: A random variable is a function that associates a point of the sample space with a real number.

Random Process/Stochastic Process: A random process or stochastic process is a family (or collection) of random variables.

For example, if a die with six faces numbered 1, 2, ..., 6 is thrown, the set S is a sample space where the set of all possible outcomes is $S = \{1, 2, 3, 4, 5, 6\}$. A function X that assigns to an outcome or sample point, the number written on it, is the random variable. The event that an odd number is observed corresponds to the set $\{1, 3, 5\}$ which is a subset of S.

Let us take more examples of stochastic process.

- i) If $n \geq 1$, Suppose X_n is the outcome of the n^{th} throw. Then $\{X_n, n \geq 1\}$ is a collection of random variables, such that for a distinct value of n, one gets a distinct random variable X_n . The sequence $\{X_n, n \geq 1\}$ constitutes a random (stochastic) process called the Bernoulli process.

- ii) Let X_n represents the number of sixes in the first n throws. For each value of $n=1, 2, \dots$, we get a distinct Binomial random variable. The sequence $X_n: \{X_n, n \geq 1\}$, which gives a collection of random variables, is a random or stochastic process.
- iii) Suppose a telephone call is received at switchboard. Let X_t be the random variable, which represents the number of incoming calls in an interval $(0, t)$. Then X_t is a random variable and the family $\{X_t, t \in T\}$ constitutes a stochastic process, where T is the interval $0 \leq t \leq \infty$.

6.2.1. Poisson Process

Suppose $X(t)$ represent the maximum temperature at a particular place. Here we deal with discrete state, continuous time stochastic processes. The Poisson process is one of the representatives of this type of stochastic processes. The Poisson process may be explain as follows:

Let E be a random event such as (i) incoming telephone calls at a switchboard, (ii) arrival of patients for treatment at a clinic, (iii) occurrence of accidents at a certain place.

We consider the total number $N(t)$ of occurrences of an event E in an interval of time ' t '. Suppose $P_n(t)$ is the probability that the random variable $N(t)$ assumes the value n , i.e.,

$$P_n(t) = P[N(t)=n] \quad \dots 1.$$

For $n=0, 1, 2, 3, \dots$. We have,

$$\sum_{n=0}^{\infty} P_n(t) = 1, \text{ for each fixed } t. \quad \dots 2$$

We can thus say from equation (2) that $P_n(t)$ is a probability mass function of the random variable $N(t)$ and the family of random variables $[N_t, t > 0]$ is a stochastic process. From our earlier discussion, you may understand that this family is continuous parameter (in this case, time) stochastic process with a discrete state space. This is called a Poisson process. Under certain conditions, $N(t)$ follows a Poisson distribution with mean λt (λ is being constant). This holds for most practical situations.

6.2.1.1. Assumptions in Poisson Process

- Events must be independent, in other words the number of the customers which is arrive in disjoint time intervals are statistically independent e.g. the number of goals scored by a team should not make the number of goals scored by another team more or less likely and the mean number of goals scored is assumed to be the same for all teams.
- The probability of two events occurring between time t and $t + \Delta t$ is $O(\Delta t)$, i.e. negligible. Thus.,

$$P_0(\Delta t) + P_1(\Delta t) + O(\Delta t) = 1$$

- The probability that event E occurs between time t and Δt is equal to $\lambda \Delta t + O(\Delta t)$. Thus, $P_1(\Delta t) = \lambda \Delta t + O(\Delta t)$,

Under the assumptions stated above, $N(t)$ follows a Poisson distribution with mean λt , i.e., $P_n(t)$ is given by

$$P_n(t) = \frac{e^{-\lambda t} (\lambda t)^n}{n!} \quad n=0, 1, 2, 3, \dots$$

To formulate a queueing model, we have to specify the assumed form of the probability distributions of both inter arrival times and service e times. You have learnt that inter- arrival times follow the Poisson distribution. Similarly, most of the times, the service times in a queueing system follow the exponential distribution. Hence, the exponential distribution is the most important distribution in queueing theory

6.2.2. Markov Process

Around the beginning of the 20th century, the methodology discussed here was developed by the Russian mathematician. A.A. Markov. The Markov Process forms a sub-class of the set of all random processes. It is a sub class with enough simplifying assumptions to make them easy to handle.

A stochastic (or random) process is called a Markov process if the occurrence of a future state depends on the immediately preceding state and only on it. If, for all t_n, t_{n-1}, \dots, t_0 , satisfying $t_n > t_{n-1} > \dots > t_0$ and non- negative integer j , the family of random variables $\{X(t), t \geq 0\}$ is said to be a Markov process if it satisfy the Markovian property:

$$P[X(t_n)=j_n/X(t_{n-1})=j_{n-1}, \dots, X(t_0)=j_0]=P[X(t_n)=j_n/X(t_{n-1})=j_{n-1}],$$

The process has the Markov property and is called a continuous time Markov process.

6.2.3. Birth and Death Process

The arrival process assumes that the customers arrive at the queueing system and never leave it. Such a system is known as pure birth process. On the other hand, the departure process assumes that no customer joins the system while service is continued for those who are already in the system. If at time $t=0$, N is greater than or equal to one i.e. at starting time number of customers is N . Since service is being provided at the rate of μ , therefore customers leave the system at the rate of μ after being serviced. This type of process is known as pure death process.

Let the state at time t would correspond to the number of arrivals by time t . $P_n(t)$ denotes the probability that there are n customers in the system at any time t , both waiting and being served. Arrivals can be considered as births. If the system is in state E_n and arrival occurs, the state is changed to E_{n+1} . Similarly, a departure can be looked upon as death. A departure occurring while the system is in state E_n changes the system to the state E_{n-1} . This type of process is generally referred to as a birth – death process.

6.2.3.1. Assumptions in the Birth-Death Process

The assumptions of the birth-death process are as follows:

- If the system is in state E_n , the current probability distribution of the time t until the next arrival (birth) is exponential with parameter λ_n , where $n = 0, 1, 2, \dots$
- If the system is in state E_n , the current probability distribution of the time t until the next departure or service completion (death) is exponential with parameter μ_n , where $n = 0, 1, 2, \dots$
- Only one birth or death can occur in a small interval of time, i.e., Δt .

6.3. FUNDAMENTAL STRUCTURE OF A QUEUING SYSTEM

Queuing theory is related with the mathematical study of queues or waiting lines, a queue is formed when there are customers who require some sort of services and the current demand for a service exceeds the current capacity to provide the service.

Generally, the customer's arrival and their service time are not known in advance or can't be predicted accurately. Since arrival-departure process are random. So, queuing models developed to reduce waiting time/excessive costs and work for maintaining balance between service capacity and waiting time.

A simple queuing system can be described as follows:

1. Input or arrival process of customers
2. Service mechanism (or process)
3. Queue discipline

The fundamental structure of a queueing system shown in the figure given below



Now we explain all the components of queueing system.

1. Input or Arrival Process of Customers

The rate at which the customers arrive at the service facility is determined by the arrival process. An input/arrival process can be defined completely by its size, the arrival time distribution, and the attitude of the customers. We describe these, in brief.

- i) **Size:** It may be finite or infinite according as the arrival rate is affected or not affected by the number of customers in the service system.
- ii) **The arrival time distribution:** Mostly, the arrival time distribution is approximated by Poisson distribution.
- iii) **Customer or arrivals behavior:**
 - A customer who stay in the system until served no matter how much he has to wait for service. Such a customer is called Patient Customer.
 - The customer who wait for a certain time in the queue and leave the system without getting service. This kind of customer is known as impatient or reneging behavior.
 - If a customer before joining the system get discouraged by seeing the number of customers already in the queue is too large and does not join the queue. This behavior of the customer is called Balking behavior.

- Customers who move from one queue to another because they think that their queue is moving slower, the behavior of the customer is known as queue jockeying.

Remark: Generally, it is assumed that the customers arrive into the system one at a time. But, sometimes, customers may arrive in groups and such arrival is called Bulk arrival.

2. Service Mechanism (or Process)

Service time distributions are generally exponential distributions. It may be any one of the following types:

1. Single channel facility
2. One queue-several station facilities
3. Several queues-one service station
4. Multi-channel facility and
5. Multistage channel facility

3. Queue Discipline

Queue discipline is the order or the manner in which the service station selects the next customer from the waiting line to be served. There are many ways in which a customer to be selected for service. Some of these are described as follows:

- i) FIRST IN, FIRST OUT (FIFO) or in other words First Come First Served (FCFS);
- ii) Last in First Out (LIFO); and
- iii) Service in Random Order (SIRO)

Throughout the chapter, we consider the FCFS queue discipline.

6.4. OPERATING CHARACTERISTICS OF A QUEUEING SYSTEM

For studying queuing system, we have to set up a set of equations. We have to solve these equations to determine the operating characteristics (or performance measures). There are two types of solutions of these equations: (i) **Transient** (ii) **Steady state**.

Transient solutions: The time dependent solutions are known as transient solutions.

Steady state solutions: These solutions are independent of time and represent the probability of the system being in a particular state in the long run.

Note: Here we shall be analysing the system under the steady state condition.

6.4.1. Operating characteristics/ Performance Measures

Performance Measures of a queuing system are determined by two statistical properties, namely, the probability distribution of inter-arrival times and the probability distribution of service times.

Some of the operating characteristics/performance measures of any queueing system that are of general interest for analysing the system are listed below:

1. L_s : The average number of customers in the queuing system (those waiting to be served and those being served).
2. W_s : The average time each customer spends in the queuing system from entry into the queue to completion of the service (the time spent waiting in the queue and during the service).
3. L_q : The average number of customers in the queue waiting to get service (this excludes customers undergoing service).
4. W_q : The average time each customer spends in the queue waiting to get service (this excludes customers time spent during the service).
5. **Service idle time**: The relative frequency with which the service system is idle.

Now we describe the equations for each of the operating characteristics listed above under steady state solutions. By definition of expectation, we have

$$L_s = E(n) = \sum_{n=1}^{\infty} nP_n \quad (\text{Expected number of customers in the system}) \quad (1)$$

$$L_q = \sum_{n=c+1}^{\infty} (n-c)P_n \quad (\text{Expected number of customers in queue}) \quad (2)$$

where there are c parallel servers so that c customers can be served simultaneously with λ arrival rates for those who join the system, we have

$$L_s = \lambda W_s \quad (3)$$

and

$$L_q = \lambda W_q \quad (4)$$

If μ is the service rate, the expected service time is $\frac{1}{\mu}$ and we have

$$W_s = W_q + \frac{1}{\mu} \quad (5)$$

Where, W_s denotes expected waiting time in system and W_q represents expected waiting time in queue

Multiplying both sides of equation (5) with λ , we get

$$\begin{aligned} \lambda W_s &= \lambda W_q + \frac{\lambda}{\mu} \\ \Rightarrow L_s &= L_q + \frac{\lambda}{\mu} \quad (\text{By (3) and (4)}) \end{aligned} \quad (6)$$

6.4.2. Classification of Queueing Models

Generally, queueing models are described by five symbols such as a/b/c: d/e or a/b/c/d/e. The first symbol 'a' describes the arrival process. The second symbol 'b' describes the service time distribution. The third

symbols 'c' stands for the number of servers. The symbols 'd' and 'e' stand for the system capacity and queue discipline respectively.

First three symbols i.e. a/b/c in the above notation were described by D. Kendall in 1953. Later, A. Lee in 1966 added the fourth (d) and fifth (e) to the Kendall notation.

Remark: The fifth symbol (e) from the notation can be omitted if the system has FCFS queue discipline and for this case it can be described as a/b/c/d. Moreover, if the system has infinite capacity with FCFS queue discipline then we can use simply a/b/c or a/b/c:∞/FCFS for describing the queueing system.

Note: If arrivals follow Poisson distribution and departures follow exponential distribution, symbol M is used in place of 'a' and 'b' e.g. suppose there is a single server then we can use the notation M/M/1 (here we consider infinite capacity and FCFS queue discipline).

6.5. M/M/1 OR M/M/1: ∞/FCFS QUEUEING MODEL

This model deals with a queueing system having a single server channel and no limit on system capacity with Poisson input and exponential distribution for services while the customers are served on a "First Come First Served" basis. For the given model, arrivals follow Poisson distribution and hence inter-arrival times have an exponential distribution. The service time for this model follows an exponential distribution.

If λ and μ denote the average arrival rate and average service rate respectively then $1/\lambda$ and $1/\mu$ show mean arrival time and mean service time correspondingly. Then the ratio $\rho = \frac{\lambda}{\mu}$ is called the traffic

intensity or the utilisation factor. It is the measure of the degree to which the capacity of the service station is utilised. For example, if customer arrives at a rate of 10 per minute and the service rate 20 per minute, the utilisation of the service capacity is $\frac{10}{20} = 50\%$, i.e., the service facility is kept busy 50% of the time and remains idle 50% of the time.

6.5.1. Arrival-Departure Equations for the M/M/1 Queuing Model

OR

System of Differential-Difference Equations

If $n \geq 1$, the probability that the system has n customers at time $(t + \Delta t)$ can be expressed as:

$$P_n(t + \Delta t) = [\{\text{Probability of } n - 1 \text{ customers in the system at time } t \text{ and (1 arrival, no departure in time } \Delta t \text{ or 2 arrivals, one departure in time } \Delta t \text{ or 3 arrivals, two departure in time } \Delta t \text{ or } \dots)\} + \{\text{Probability of } n - 1 \text{ customers in the system at time } t \text{ and (2 arrivals, no departure in time } \Delta t \text{ or 3 arrivals, one departure in time } \Delta t \text{ or 4 arrivals, two departures in time } \Delta t \text{ or } \dots)\} + \{\text{Probability of } n - 3 \text{ customers in the system at time } t \text{ and (3 arrivals, no departure in time } \Delta t \text{ or 4 arrivals, one departure in time } \Delta t \text{ or 5 arrivals, two departures in time } \Delta t \text{ or } \dots)\} + \dots] + [\{\text{Probability of } n + 1 \text{ customers in the system at time } t \text{ and (1 departure, no arrival in time } \Delta t \text{ or 2 departures, one arrival in time } \Delta t \text{ or 3 departures, two arrivals in time } \Delta t \text{ or } \dots)\} + \{\text{Probability of } n + 2 \text{ customers in the system at time } t \text{ and (2 departures, no arrivals in time } \Delta t \text{ or 3 departures, one arrival in time } \Delta t \text{ or } \dots)\} + \{\text{Probability of } n + 3 \text{ customers in the system$$

at time t and (3 departures, no arrival in time Δt or ...) + ...] + [{Probability of n customers in the system at time t and (no arrival, no departure in time Δt or 1 arrival, one departure in time Δt or 2 arrivals, two departures in time Δt or ...)]

$$\begin{aligned}
&= P_{n-1}(t) \left[\{ \lambda \Delta t + O(\Delta t) \} \{ 1 - \mu \Delta t + O(\Delta t) \} + \text{The terms equal to } O(\Delta t) \right] \\
&+ P_{n-2}(t) \left[\{ O(\Delta t) \} \{ 1 - \mu \Delta t + O(\Delta t) \} + \text{The terms equal to } O(\Delta t) \right] \\
&+ P_{n-3}(t) O(\Delta t) + \text{The terms equal to } O(\Delta t) \\
&+ P_{n+1}(t) \left[\{ (\mu \Delta t + O(\Delta t)) \} \{ 1 - \lambda \Delta t + O(\Delta t) \} + \text{The terms equal to } O(\Delta t) \right] \\
&+ P_{n+2}(t) \left[\{ O(\Delta t) \} \{ 1 - \lambda \Delta t + O(\Delta t) \} + \text{The terms equal to } O(\Delta t) \right] \\
&+ P_{n+3}(t) O(\Delta t) + \text{The terms equal to } O(\Delta t) \\
&+ P_n(t) \left[\{ 1 - \mu \Delta t + O(\Delta t) \} \{ 1 - \lambda \Delta t + O(\Delta t) \} \right. \\
&\left. + \{ \mu \Delta t + O(\Delta t) \} \{ \lambda \Delta t + O(\Delta t) \} + \text{The terms which ultimately become equal to } O(\Delta t) \right]
\end{aligned}$$

Notice that $\Delta t \cdot O(\Delta t)$, $O(\Delta t) \cdot O(\Delta t)$, $O(\Delta t) + O(\Delta t)$, $O(\Delta t) - O(\Delta t)$, $\Delta t \cdot \Delta t$ are very small and may be taken as equal to $O(\Delta t)$

We deal with the case $n = 0$ separately because there cannot be any possibility of less than zero customers. However, in the above equation, the case of less than n customers is included.

The probability of 0 (Zero) customers in the system at time $t + \Delta t = [\{ \text{Probability of 1 customer in the system at time } t \text{ and (1 departure, no customer/arrival in time } \Delta t \text{ or 2 departure, 1 arrivals in time } \Delta t \text{ or 3 departures, two arrivals in time } \Delta t) \} + \{ \text{Probability of 2 customers in the system at time } t \text{ and (2 departures, no arrival in time } \Delta t \text{ or 3 departures, one arrival in time } \Delta t \text{ or 4 departures, 2 arrivals in time } \Delta t) \} + \{ \text{Probability of 3 customers in the system at time } t \text{ and (3 departures, no arrival in time } \Delta t \text{ or 4 departures, one arrival in time } \Delta t \text{ or 5 departures, two arrivals in time } \Delta t) \} + \dots] + [\{ \text{Probability of } O \text{ customers in the system at time } t \text{ and (no arrival, no departure in time } \Delta t \text{ or 1 arrival, one departure in time } \Delta t \text{ or 2 arrivals, 2 departures in time } \Delta t) \}]$

$$\begin{aligned}
&= P_1(t) \left[\{ \mu \Delta t + O(\Delta t) \} \{ 1 - \lambda \Delta t + O(\Delta t) \} + \text{The terms which ultimately become equal to } O(\Delta t) \right] \\
&+ P_2(t) \left[\{ O(\Delta t) \} \{ 1 - \lambda \Delta t + O(\Delta t) \} + \text{The terms which ultimately become equal to } O(\Delta t) \right] \\
&+ P_3(t) O(\Delta t) + \text{The terms which ultimately equal to } O(\Delta t)
\end{aligned}$$

$P_0(t)$ [{Probability of no departure which 1 as there is no customer in the system and hence no chance of departure}]

$\{1 - \lambda\Delta t + O(\Delta t)\} \{ \mu\Delta t + O(\Delta t) \} \{ \lambda\Delta t + O(\Delta t) \} +$ The terms which ultimately become equal to $O(\Delta t)$]

Therefore, the arrival-departure equations for the M/M/1 Queuing model are:

$$\begin{aligned}
 P_n(t + \Delta t) &= P_{n-1}(t) [\lambda\Delta t + O(\Delta t)] + P_{n+1}(t) [\mu\Delta t + O(\Delta t)] \\
 &+ P_n(t) [\lambda\Delta t + O(\Delta t)] [\mu\Delta t + O(\Delta t)] \\
 &+ P_n(t) [1 - \lambda\Delta t] [1 - \mu\Delta t] + O(\Delta t), n \geq 1 \\
 P_0(t + \Delta t) &= P_1(t) [\mu\Delta t + O(\Delta t)] + P_0(t) [\lambda\Delta t + O(\Delta t)] \\
 &+ P_0(t) [1 - \lambda\Delta t] \cdot 1 + O(\Delta t), n = 0
 \end{aligned} \tag{1}$$

$$\begin{aligned}
 P_n(t + \Delta t) &= P_{n-1}(t) \lambda\Delta t + P_{n+1}(t) \mu\Delta t \\
 &+ P_n(t) [1 - \lambda\Delta t - \mu\Delta t] + O(\Delta t), n \geq 1 \\
 P_0(t + \Delta t) &= P_1(t) \mu\Delta t + P_0(t) [1 - \lambda\Delta t] + O(\Delta t)
 \end{aligned} \tag{2}$$

Dividing equation (2) by Δt and taking the limit as Δt tends to zero, we have

$$\begin{aligned}
 \lim_{\Delta t \rightarrow 0} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} &= \lambda P_{n-1}(t) + \mu P_{n+1}(t) - (\lambda + \mu) P_n(t) \\
 \Rightarrow P_n^1(t) &= \lambda P_{n-1}(t) + \mu P_{n+1}(t) - (\lambda + \mu) P_n(t)
 \end{aligned} \tag{3}$$

and

$$P_0^1(t) = -\lambda P_0(t) + \mu P_1(t) \tag{4}$$

When steady state, i.e., the equilibrium state, is reached, $P_n^1(t)$ becomes independent of time, say p_n , and the rate of its change with respect to time becomes zero, i.e.,

$$P_n^1(t) = 0.$$

Therefore, the **steady-state solution** is given by

$$\begin{aligned}
 \lambda P_{n-1} + \mu P_{n+1} - (\lambda + \mu) P_n &= 0, n \geq 1 \\
 \mu P_1 &= \lambda P_0
 \end{aligned} \tag{5}$$

Now, from equation (5), we have

$$\mu P_{n+1} - \lambda P_n = \mu P_n - \lambda P_{n-1} \quad (6)$$

This implies that

$$\mu P_n - \lambda P_{n-1} = \mu P_{n-1} - \lambda P_{n-2} \text{ (changing } n \text{ to } n-1)$$

$$\mu P_{n-1} - \lambda P_{n-2} = \mu P_{n-2} - \lambda P_{n-3} \text{ (again, changing } n \text{ to } n-1)$$

$$\mu P_{n-2} - \lambda P_{n-3} = \mu P_{n-3} - \lambda P_{n-4} \text{ (again, changing } n \text{ to } n-1)$$

...

...

...

$$\mu P_2 - \lambda P_1 = \mu P_1 - \lambda P_0$$

But $\mu P_1 - \lambda P_0 = 0$ (from (5))

Therefore, $\mu P_{n+1} - \lambda P_n = \mu P_n - \lambda P_{n-1}$

$$= \mu P_{n-1} - \lambda P_{n-2}$$

⋮

$$= \mu P_1 - \lambda P_0 = 0$$

$$\Rightarrow P_n = \frac{\lambda}{\mu} P_{n-1}$$

$$= \frac{\lambda}{\mu} \left(\frac{\lambda}{\mu} P_{n-2} \right) \left(\because \mu P_{n-1} - \lambda P_{n-2} = 0 \Rightarrow P_{n-1} = \frac{\lambda}{\mu} P_{n-2} \right)$$

$$= \frac{\lambda}{\mu} \frac{\lambda}{\mu} \cdot \left(\frac{\lambda}{\mu} P_{n-3} \right) \left(\because \mu P_{n-2} - \lambda P_{n-3} = 0 \Rightarrow P_{n-2} = \frac{\lambda}{\mu} P_{n-3} \right)$$

⋮

$$= \frac{\lambda \cdot \lambda \cdot \lambda \dots \lambda (n \text{ times})}{\mu \cdot \mu \cdot \mu \dots \mu (n \text{ times})} P_0 = \frac{\lambda^n}{\mu^n} P_0$$

$$= \left(\frac{\lambda}{\mu} \right)^n P_0 = P^n P_0.$$

Since $\sum_{n=0}^{\infty} P_n = 1$, it follows that

$$\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} P_0 P^n = 1 \quad (\because P_n = P \dots \dots \dots)$$

$$\Rightarrow P_0(1 + HP + P^2 + \dots + P^n + \dots) = 1$$

$$\Rightarrow P_0 \left(\frac{1}{1-P} \right) = 1$$

$$\Rightarrow P_0 = 1 - P \text{ Where } P = \frac{\lambda}{\mu}$$

Therefore, the probability of n customers (units) in the system is given by

$$P_n = (1 - P)P^n, n \geq 0$$

Model \rightarrow **M/M/1 or M/M/1: ∞ /FCFS**

Probability of n customers in the system is given by

$$P_n = (1 - \rho)\rho^n, n \geq 1$$

Probability of zero customers (units) in the queue/system is given by $P_0 = 1 - \rho$

6.5.2. Operating Characteristics or Measures of Performances of the Model M/M/1

1. The average number of customers (units) in the system is given by

$$L_s = E(n) = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1 - \rho)\rho^n = (1 - \rho) \sum_{n=0}^{\infty} n\rho^n \quad \dots(*)$$

Now, consider $\sum_{n=0}^{\infty} n\rho^n = 0 + \rho + 2\rho^2 + 3\rho^3 + \dots$

$$\text{Let } S = \sum_{n=0}^{\infty} n\rho^n = \rho + 2\rho^2 + 3\rho^3 + \dots$$

$$\Rightarrow \rho S = \rho^2 + 2\rho^3 + 3\rho^4 + \dots$$

$$\Rightarrow S - \rho S = (\rho + 2\rho^2 + 3\rho^3 + \dots) - (\rho^2 + 2\rho^3 + 3\rho^4 + \dots)$$

$$\Rightarrow S(1 - \rho) = \rho + \rho^2 + \rho^3 + \dots$$

$$\Rightarrow S(1 - \rho) = \frac{\rho}{1 - \rho}$$

$$\Rightarrow S = \frac{\rho}{(1 - \rho)^2}$$

Thus from (*), we have $L_s = (1 - \rho)S = \frac{(1 - \rho)\rho}{(1 - \rho)^2} = \frac{\rho}{1 - \rho}$

Now, ρ is the traffic intensity or utilisation factor, where $\rho = \frac{\lambda}{\mu}$

$$\text{Thus, } L_s = \frac{\rho}{1-\rho} = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda} \quad (1)$$

Where, λ denotes the average arrival rate and μ denote the average service rate.

2. The average queue length (Expected number of customers) is L_q

$$\begin{aligned} \text{And } L_q &= L_s - \text{Traffic intensity} \\ &= \frac{\lambda}{\mu-\lambda} - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu-\lambda)} \end{aligned} \quad (2)$$

3. The average time an arrival spends in the system (Expected waiting time in system) is

$$W_s = \frac{L_s}{\lambda} = \frac{\lambda}{\lambda(\mu-\lambda)} = \frac{1}{\mu-\lambda} \quad (\text{from (1)})$$

$$\text{Thus } W_s = \frac{1}{\mu-\lambda} \quad (3)$$

4. The average waiting time of an arrival in the queue is W_q .

$$\text{And } W_q = \frac{L_q}{\lambda} = \frac{\lambda^2}{\lambda\mu(\mu-\lambda)} = \frac{\lambda}{\mu(\mu-\lambda)} \quad (4)$$

from (4), we can also obtain the relation

$$W_q = W_s - \frac{1}{\mu}$$

5. The probability that the number of customers (units) waiting in the queue and the number of units being serviced is greater than k is

$$P[n > k] = \rho^{k+1} \quad (5)$$

6. The probability of having a queue i.e. $1 - P_0$

$$\text{Thus } 1 - P_0 = 1 - (1 - \rho) = \rho \quad (6)$$

Now, we explain the M/M/1 queuing model with the help of following examples.

6.5.3. Example. A TV repairman finds that the time spent on his job has an exponential distribution with mean 30 minutes. If he repairs sets in the order in which these come in, and if the arrival of sets is approximately Poisson with an average rate of 10 per 8-hour day, what is the repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?

Solution: Here, we have

$$\text{Arrival rate } (\lambda) = \frac{10}{8} \text{ arrivals/hr}$$

$$\text{Service rate } (\mu) = \frac{60}{30} = 2 \text{ services/hr}$$

\therefore The probability of the repairman being idle is

$$P_0 = 1 - \rho = 1 - \frac{\lambda}{\mu} = 1 - \frac{10}{8} \times \frac{1}{2} = 1 - \frac{5}{8} = \frac{3}{8}$$

$$\text{Thus, } P_0 = \frac{3}{8}.$$

Hence the idle time in the 8-hour day = $\frac{3}{8} \times 8$ hours = 3 hours

Now, the average number of jobs that are ahead of the set (person) just brought in

= Average number of units in the system means who are in queue and at counter, i.e.

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{10}{8}}{2 - \frac{10}{8}} = \frac{10}{6} = \frac{5}{3} = 1.666$$

$$\text{Thus, } L_s = 1.67.$$

6.5.4. Example. Customers arrive at a window in a bank, according to a Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean 5 minutes. The space in front of the window including that for the serviced customers can accommodate a maximum of three customers. Other customers can wait outside this space.

- What is the probability that an arriving customer can go directly to the space in front of the window?
- What is the probability that an arriving customer will have to wait outside the indicated space?
- How long is an arriving customer expected to wait before being served?

Solution: Here, arrival rate $\lambda = 10$ per hour and service rate $(\mu) = \frac{60}{5}$ per hour = 12 per hour

$$\text{Thus, traffic intensity } (\rho) = \frac{\lambda}{\mu} = \frac{10}{\frac{60}{5}} = \frac{5}{6}$$

$$\Rightarrow \rho = \frac{5}{6}$$

(a) The probability that an arriving customer can go directly to the space in front of the window i.e. the probability of n customers in the system here $2 \geq n \geq 0$, therefore the required probability = $P_0 + P_1 + P_2$

$$\begin{aligned} &= \left(1 - \frac{5}{6}\right) + \left(1 - \frac{5}{6}\right) \frac{5}{6} + \left(1 - \frac{5}{6}\right) \left(\frac{5}{6}\right)^2 \\ &= \frac{1}{6} + \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot \frac{25}{36} \approx 0.42 \end{aligned}$$

(b) The probability that an arriving customer will have to wait outside the indicated space, i.e. the required probability = $P_3 + P_4 + P_5 + \dots$

$$\begin{aligned} &= 1 - (P_0 + P_1 + P_2) \\ &= 1 - 0.42 = 0.58 \end{aligned}$$

(c) Expected waiting time before being served

$$= W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{10}{12(12 - 10)} = 0.417 \text{ hours}$$

6.5.5. Exercises.

1. Customers arrive at a box office window being served by a single individual according to a Poisson input process with a mean rate of 30 per hour. The time required to serve a customer has an exponential distribution with a mean of 90 seconds. Find the average waiting time of a customer in the queue.

$$\begin{aligned} \text{Solution. } W_q &= \frac{\lambda}{\mu(\mu - \lambda)} = \frac{30}{40(40 - 30)} = \frac{3}{10} \text{ hr} \\ &= \frac{3}{4} \times 60 \text{ minutes} = 4.5 \text{ minutes} \end{aligned}$$

2. A fertilizer company distributes its products by trucks loaded at its only loading station. Both company trucks and contractor's truck are used for this purpose. It was noticed that on an average on truck arrived every 5 minutes and the average loading time was 3 minutes. Forty percent of the trucks belong to the contractor. Determine the expected waiting time of contractor's trucks per day.

Solution. The expected waiting time of contractor's truck per day = 8.64 hrs.

3. In a railway marshalling yard, goods train arrives at a rate of 36 trains per day. Assuming that inter-arrival time follows exponential distribution and the service time distribution is also exponential with an average of 30 minutes, calculate the following:

- i. The mean line length
- ii. The probability that the queue size exceeds 10.
- iii. If the input increases to an average of 42 per day, what will the change in (a) and (b) be?

Solution. (i) Mean length of a system

$$L_s = 3 \text{ trains 1}$$

(ii) Probability that the queue size exceeds 10 = probability that the system size exceeds 11

$$\text{i.e., } P(n > 11) = (0.75)^{12}$$

$$\text{c) } L_s = \lambda' / \mu - \lambda' = 42/48 - 42 = 7 \text{ trains}$$

And the probability that the queue size exceeds 10 is $(7/8)^{12}$

6.6. CHECK YOUR PROGRESS

1. What is Queueing Theory?
2. Explain what do you mean by Poisson process.
3. Describe birth and death process.
4. Explain briefly fundamental structure of Queueing System.
5. Write a note on operating characteristics of Queueing System.
6. Explain classification of Queueing System.
7. Derive the arrival-departure equation for M/M/1 Queueing Model.
8. State operating characteristics of M/M/1 Queueing Model.

6.7. SUMMARY

In this chapter, we have introduced the concept of queueing theory. It has been observed that for understanding queueing system, we have to familiar with some concepts namely Stochastic Process, Poisson Process, and Birth-Death Process along with some distributions such as Poisson distribution and exponential distribution. We explained the basic component, fundamental structure and operating characteristics of a queueing system. Also, we discussed the M/M/1, queueing models.

7

QUEUEING MODELS

Structure

7.1. Introduction.

7.2. M/M/C or M/M/C: ∞ /FCFS Queueing Model.

7.3. M/M/1/K Queueing Model.

7.4. M/M/C/K Queueing Model.

7.5. Check Your Progress.

7.6. Summary.

7.1. INTRODUCTION

Queueing theory is the mathematical study of waiting lines or queues and considered a branch of operations research because the results are often used when making business decisions about the resources needed to provide a service. A queueing model is constructed for predicting queue lengths and waiting time.

In the chapter 6, we have discussed the basic concept of queueing theory which would help us in understanding the techniques of queueing models. We studied about the fundamental structure, operating characteristics and classification of queueing models. Also, we described M/M/1 queueing model and its applications. In the present chapter, we shall describe some other queueing models with single and multiple servers namely M/M/C, M/M/1/k and M/M/C/k queueing models.

7.1.1. Objectives. The objective of these contents is to explain some important queueing models with single and multiple servers to the reader such as:

- M/M/C Queueing Model.

- M/M/1/k Queueing Model.
- M/M/C/k Queueing Model.

7.2. M/M/C OR M/M/C: ∞ /FCFS QUEUEING MODEL

This model deals with a queueing system having a multiple server channel and no limit on system capacity with Poisson input and exponential distribution for services while the customers are served on a “First Come First Served” basis. We consider a queue with Poisson input (having arrival rate, say, λ) and with C ($1 \leq C < \infty$) parallel service channels having independently and identically distributed exponential service time distribution, each with rate, say, μ .

If there are n units in the system, and n is less than C , then, in all, n channels, are busy. If there are n ($\geq C$) in the system, then all the C channels are busy. Thus, we have a birth death model having constant arrival (birth) rate λ and state-dependent service (death) rate.

$$\mu_n = \begin{cases} n\mu, & n = 0, 1, 2, \dots, C \\ C\mu, & n = C+1, C+2, \dots \end{cases}$$

Hence, the steady state probability exist if

$$\rho = \frac{\lambda}{C\mu} < 1 \text{ and are given by}$$

$$P_n = \begin{cases} \frac{\lambda \cdot \lambda \cdot \dots \lambda}{\mu \cdot 2\mu \cdot \dots n\mu} P_0, & 0 \leq n < C \\ \frac{\lambda \cdot \lambda \cdot \dots \lambda}{(\mu \cdot 2\mu \cdot \dots C\mu)(C\mu \cdot \dots C\mu)} P_0, & n \geq C \end{cases}$$

where, the term $(c\mu \cdot \dots c\mu)$ is repeated $(n-c)$ times.

$$= \begin{cases} \left(\frac{\lambda}{\mu} \right)^n \frac{P_0}{n!}, & 0 < n \leq C \\ \left(\frac{\lambda}{\mu} \right)^n \frac{P_0}{C! \cdot C^{n-C}}, & n \geq C \end{cases}$$

$$\text{Now, } \sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow \sum_{n=0}^{C-1} P_n + \sum_{n=C}^{\infty} P_n = 1 \Rightarrow P_0 \left[\sum_{n=0}^{C-1} \frac{\left(\frac{\lambda}{\mu} \right)^n}{n!} + \sum_{n=0}^{C-1} \frac{\left(\frac{\lambda}{\mu} \right)^n}{C! \cdot C^{n-C}} \right] = 1$$

$$\Rightarrow P_0 \left[\sum_{n=0}^{C-1} \frac{C^n \rho^n}{n!} + \sum_{n=C}^{\infty} \frac{C^C \cdot \rho^n}{C!} \right] = 1, \text{ where } \rho = \frac{\lambda}{C\mu}$$

$$\Rightarrow P_0 \left[\sum_{n=0}^{C-1} \frac{C^n \rho^n}{n!} + \sum_{n=C}^{\infty} \frac{C^C \rho^n}{C!} \right]^{-1}$$

$$= \left[\sum_{n=0}^{C-1} \frac{C^n \rho^n}{n!} + \frac{C^C}{C!} \rho^C \frac{1}{1-\rho} \right]^{-1}$$

7.2.1. Operating characteristics/Performance measures of M/M/C Model

In chapter 6, we have already defined the meaning of performances measures such as L_q, L_s, W_q and W_s , here we only evaluate mathematical expression for all these as given below:

$$1. \quad L_q = \sum_{n=C}^{\infty} (n-C) P_n$$

$$= \sum_{j=1}^{\infty} j P_{C+j}, \text{ Now put } n-C=j, \text{ we get}$$

$$= P_0 \sum_{j=1}^{\infty} \frac{j \left(\frac{\lambda}{\mu} \right)^{C+j}}{C! C^j}$$

$$= \frac{P_0 \left(\frac{\lambda}{\mu} \right)^C}{C!} \sum_{j=1}^{\infty} j \rho^j$$

$$= \frac{P_0 \left(\frac{\lambda}{\mu} \right)^C}{C!} \frac{\rho}{(1-\rho)^2}$$

$$\left[\because \rho \sum_{j=1}^{\infty} j \rho^{j-1} = \rho \sum_{j=1}^{\infty} \frac{d}{d\rho} (\rho^j) = \rho \frac{d}{d\rho} \left(\frac{\rho}{1-\rho} \right) = \rho \left[\frac{(1-\rho) - \rho(-1)}{(1-\rho)^2} \right] \right]$$

$$2. \quad L_s = L_q + \frac{\lambda}{\mu} \text{ (Expected number of customers in the system)}$$

3. Expected waiting time in the queue

$$W_q = \frac{1}{\lambda} \cdot L_q$$

4. Expected waiting time in the system

$$W_s = W_q + \frac{1}{\mu} \text{ or } W_s = \frac{L_s}{\lambda}$$

The following examples illustrate M/M/C queueing Model.

7.2.2. Example. A petroleum company is considering expansion of its unloading facility at its refinery. Due to random of variations in weather, loading delays and other factors, ships arriving at the refinery to unload crude oil arrive at an average rate of 5 ships per week. Service rate on an average is 10 ships per week. Assuming Poisson arrivals and exponential service distributions, find

- (a) The average time a ship must wait before beginning to deliver its cargo to the refinery;
- (b) If a second berth is rented, what will be the average number of ships waiting before being unloaded?
- (c) What would be the average time a ship would wait before being unloaded with two berths?
- (d) What is the average number of idle berths at any specified time?

Solution: (a) Here, we have the case of M/M/1 model

$$\lambda = 5 \text{ per week, } \mu = 10 \text{ per week}$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{1}{10} \text{ week}$$

(b) For this case, we have M/M/2 model

$$\text{Here } \rho = \frac{\lambda}{C\mu} = \frac{5}{2 \times 10} = \frac{1}{4}$$

$$P_0 = \left[1 + \frac{\lambda}{\mu} + \frac{\left(\frac{\lambda}{\mu}\right)^2}{2!} \cdot \frac{1}{1 - \frac{1}{4}} \right]^{-1}$$

$$= \left[1 + \frac{1}{2} + \frac{1}{8} \times \frac{4}{3} \right] = \left(\frac{3}{2} + \frac{1}{6} \right)^{-1} = \left(\frac{9+1}{6} \right)^{-1} = \frac{3}{5}$$

$$L_q = \frac{P_0 \left(\frac{\lambda}{\mu}\right)^c \rho}{C!(1-\rho)^2} = \frac{1}{30}$$

$$(c) W_q = \frac{L_q}{\lambda} = 150 \text{ week}$$

$$(d) \begin{matrix} X : & 1 & 2 \\ P : & P_1 & P_0 \end{matrix}$$

Where X be the number of idle berths and probability of number of ships in the system

$$E(X) = \sum PX = P_1 + 2P_0 = 1.5$$

Hence 2 berths are idle.

7.2.3. Example. Arrival of machinists at a tool crib are considered to be Poisson distributed at an average rate 6 per hour. The length of time the machinists must remain at the tool crib is exponentially with average time of 0.05 hours.

- a) What is the probability that a machinist arriving at the tool crib will have to wait?
- b) What is the average number of machinists at the tool crib?
- c) The company will install a second tool crib when convinced that a machinist would have to spend 6 minutes in waiting and being served at the tool crib. At what rate should the arrival of machinist to the tool crib increase to justify the addition of a second crib?

Solution. a) Here the arrival rate $\lambda = 6/\text{hr}$ and the service rate $\mu = 20/\text{hr}$

Therefore, the probability of zero customers in queue is

$$p_0 = 1 - \rho \quad \text{where } \rho = \frac{\lambda}{\mu} = \frac{6}{20}$$

$$p_0 = 1 - \frac{6}{20} = \frac{14}{20} = \frac{7}{10} = 0.7$$

The probability that a machinist arriving at the tool crib will have to wait

= Probability that there is at least one machinist at the tool crib

= 1 - Probability that there is no machinist at the tool crib

$$= 1 - p_0 = 1 - 0.7 = 0.3$$

b) The average number of machinists at the tool crib is given by

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{6}{20 - 6} = \frac{6}{14} = 0.428$$

c) The company is ready to install a second tool crib when convinced that a machinist would have to spend 6 min. in waiting and being served. Let the increased arrival rate be λ' .

$$\text{Waiting time in the system} = 6 \text{ min.} = \frac{1}{10} \text{ hr.}$$

$$\text{We have } \frac{1}{\mu - \lambda} = \frac{1}{10}$$

$$\text{or } \frac{1}{20 - \lambda} = \frac{1}{10}$$

$$\text{or } 10 = 20 - \lambda'$$

$$\text{or } \lambda' = 20 - 10 = 10/\text{hr}$$

The increase is, therefore, $(10 - 6)/\text{hr} = 4/\text{hr}$.

7.2.4. Example. A repairman is to be hired to repair machines which break down at an average rate of 3 per hour. The breakdown follows a Poisson distribution. Nonproductive time of a machine is considered to cost 10 Rs per hour. Two repairmen have been interviewed of whom one is slow but charges less and the other is fast but more expensive. The slow repairman charges 5 Rs per hour and services breakdown machines at the rate of 4 per hour. The fast repairman demands 7 Rs per hour, but services breakdown machines at an average rate of 6 per hour. Which repairman should be hired?

Solution. The data given is summarized below:

Slow/less expensive Repairman:

$$\lambda = 3/\text{hr}, \mu = 4/\text{hr}, \text{ Labour cost} = 5 \text{ Rs/hr}$$

Fast/more expensive Repairman:

$$\lambda = 3/\text{hr}, \mu = 6/\text{hr}, \text{ Labour cost} = 7 \text{ Rs/hr}$$

Case of Slow/less expensive Repairman:

Cost of engaging slow repairman for 8 hours

$$\begin{aligned}
 &= \text{Breakdown cost} + \text{Labour cost for 8 hour working day} \\
 &= (\text{No. of breakdowns per hr}) \times 8 \text{ hours} \times \text{Av. Time spent in the} \\
 &\quad \text{system} \times (\text{Breakdown cost/ cost for non-productive time}) +
 \end{aligned}$$

Labour cost per hr \times 8 hours

$$= \lambda \times 8 \times w_s \times 10 + 5 \times 8$$

Since the average time spent in the system is $W_s = \frac{1}{\mu - \lambda} = \frac{1}{4 - 3} = 1$,

The total cost for engaging the slow repairman is equal to

$$3 \times 8 \times 1 \times 10 + 5 \times 8 = 240 + 40 = 280 \text{ Rs.}$$

Case of Fast/more expensive Repairman:

Cost of engaging fast repairman for 8 hours

$$\begin{aligned}
 &= \text{Breakdown cost} + \text{Labour cost in 8 hour working day.} \\
 &= (\text{No. of breakdowns per hr}) \times 8 \text{ hours} \times \text{Av. Time spent in the} \\
 &\quad \text{System} \times (\text{Breakdown cost/ cost for non-productive time}) \\
 &\quad + \text{Labour cost per hr} \times 8 \text{ hours} \\
 &= \lambda \times 8 \times W_s \times 10 + 7 \times 8
 \end{aligned}$$

Since the average time spent in the system is $W_s = \frac{1}{\mu - \lambda} = \frac{1}{6 - 3} = \frac{1}{3}$,

The total cost for engaging the Fast repairman is

$$3 \times 8 \times \frac{1}{3} \times 10 + 7 \times 8 = 80 + 56 = 136 \text{ Rs}$$

Since the total cost of engaging the fast repairman is less, we should engage the fast repairman.

7.3. M/M/1/K QUEUEING MODEL

In this model, the capacity of the system is limited, say k , hence the maximum size of a queue is k . Here, we have single service channel. The server, serving each customer according to the exponential distribution with an average of μ customers per unit of time.

Steady State Difference Equations:

The simplest way of starting this is to treat the model as a special case of birth death process, where

$$\lambda_n = \begin{cases} \lambda, & 0 \leq n < k \\ 0, & \text{elsewhere} \end{cases}$$

and $\mu_n = \mu$ for $n=1,2,3, \dots$

Now, following the similar arguments as given in M/M/1 model, we obtain

$$P_0(t + \Delta t) = P_0(t)[1 - \lambda\Delta t] + P_1(t)\mu\Delta t + O(\Delta t), \quad n = 0$$

$$P_n(t + \Delta t) = P_n(t)[1 - (\lambda + \mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)\mu\Delta t + O(\Delta t),$$

for $n = 1, 2, \dots, k - 1$ and

$$\begin{aligned} P_k(t + \Delta t) &= P_k(t)[1 - (0 + \mu)\Delta t] + P_{k-1}(t)\lambda\Delta t + 0x\mu\Delta t + O(\Delta t) \\ &= P_k(t)[1 - \mu\Delta t] + P_{k-1}(t)\lambda\Delta t + 0 \end{aligned}$$

Now, dividing above three equations by Δt and taking limit as $\Delta t \rightarrow 0$, these equations transform into

$$P_0^l(t) = -\lambda P_0(t) + \mu P_1(t) \quad \text{for } n = 0$$

$$P_n^l(t) = -(\lambda + \mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) \quad \text{for } n = 1, 2, \dots, k - 1$$

$$\text{and } P_k^l(t) = -\mu P_k(t) + \lambda P_{k-1}(t), \quad \text{for } n = k$$

Thus, the steady-state equations are:

$$\lambda P_{n-1} + \mu P_{n+1} - (\lambda + \mu)P_n = 0, \quad 1 \leq n < k \quad (1)$$

$$-\lambda P_0 + \mu P_1 = 0 \quad (2)$$

$$\mu P_k - \lambda P_{k-1}, \quad n = k \quad (3)$$

for $1 \leq n < k$, by (1) we have

$$\begin{aligned} \mu P_{n+1} - \lambda P_n &= \mu P_n - \lambda P_{n-1} \\ &= \mu P_{n-1} - \lambda P_{n-2} \quad (\text{on replacing } n \text{ by } n - 1) \end{aligned}$$

...

...

...

$$= \mu P_1 - \lambda P_0$$

Now, from (2), (3) and the above equation, we have

$$\mu P_n - \lambda P_{n-1} = 0; \quad 0 \leq n \leq k$$

$$\Rightarrow P_n = \frac{\lambda}{\mu} P_{n-1}$$

$$= \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu} P_{n-2}$$

$$= \left(\frac{\lambda}{\mu} \right)^n P_0 = \rho^n P_0, n = 0, 1, 2, \dots, k$$

But $\sum_{n=0}^k P_n = 1$

Therefore, $\sum_{n=0}^k \rho^n P_0 = 1$

$$\Rightarrow P_0 (1 + \rho + \rho^2 + \dots + \rho^k) = 1$$

$$\Rightarrow P_0 = \frac{1}{1 + \rho + \rho^2 + \dots + \rho^k}$$

$$= \begin{cases} \frac{1-\rho}{1-\rho^{k+1}}, & \text{if } \rho \neq 1 \\ \frac{1}{k+1}, & \text{if } \rho = 1 \end{cases}$$

$$\therefore P_n = \begin{cases} \frac{\rho^n (1-\rho)}{1-\rho^{k+1}} & \text{if } \rho \neq 1 \\ \frac{1}{k+1} & \text{if } \rho = 1 \end{cases}$$

7.3.1. Operating Characteristics/Performances Measures of M/M/1/K or M/M/1: K/FCFS Queuing Model

- i. Expected number of customers in the system i.e.

$$\begin{aligned} L_s &= E(n) = \sum_{n=0}^k n P_n \\ &= \begin{cases} \sum_{n=0}^k n \frac{\rho^n (1-\rho)}{1-\rho^{k+1}}, & \text{if } \rho \neq 1 \\ \sum_{n=0}^k n \cdot \frac{1}{k+1}, & \text{if } \rho = 1 \end{cases} \\ &= \begin{cases} \frac{1-\rho}{1-\rho^{k+1}} \rho \sum_{n=1}^k n \rho^{n-1}, & \text{if } \rho \neq 1 \\ \frac{1}{k+1} [1+2+3+\dots+k], & \text{if } \rho = 1 \end{cases} \end{aligned}$$

But $\sum_{n=1}^k n \rho^{n-1} = \frac{d}{d\rho} \left[\sum_{n=1}^k \rho^n \right]$

$$\begin{aligned}
&= \frac{d}{d\rho} (\rho + \rho^2 + \rho^3 + \dots + \rho^k) = \frac{d}{d\rho} \left(\frac{\rho(1-\rho^k)}{1-\rho} \right) \\
&= \frac{(1-\rho) \{ \rho(-k\rho^{k-1}) + (1-\rho^k) \} - \rho(1-\rho^k)(-1)}{(1-\rho)^2} \\
&= \frac{-k\rho^k}{1-\rho} + \frac{(1-\rho^k)(1-\rho+\rho)}{(1-\rho)^2} = \frac{1-\rho^k}{(1-\rho)^2} - \frac{k\rho^k}{1-\rho} \\
\therefore L_s &= \begin{cases} \left[\frac{1-\rho}{1-\rho^{k+1}} \left[\frac{\rho(1-\rho^k)}{(1-\rho)^2} - \frac{k\rho^{k+1}}{1-\rho} \right] \right], & \text{if } \rho \neq 1 \\ \frac{k}{2}, & \text{if } \rho = 1 \end{cases} \\
&= \begin{cases} \frac{\rho}{1-\rho^{k+1}} \left[\frac{(1-\rho^k)}{1-\rho} - \frac{k\rho^k}{1} \right], & \text{if } \rho \neq 1 \\ \frac{k}{2}, & \text{if } \rho = 1 \end{cases} \\
&= \begin{cases} \frac{\rho}{1-\rho^{k+1}} \left[\frac{1-\rho^{k+1} + \rho^{k+1} - \rho^k}{1-\rho} - k\rho^k \right], & \text{if } \rho \neq 1 \\ \frac{k}{2} & \end{cases} \\
&= \begin{cases} \frac{\rho}{1-\rho^{k+1}} \left[\frac{1-\rho^{k+1}}{1-\rho} + \frac{\rho^{k+1} - \rho^k - k\rho^k + k\rho^{k+1}}{1-\rho} \right], & \text{if } \rho \neq 1 \\ \frac{k}{2}, & \text{if } \rho = 1 \end{cases} \\
&= \begin{cases} \frac{\rho}{1-\rho^{k+1}} \left[\frac{1-\rho^{k+1}}{1-\rho} + \frac{-\rho^k(1-\rho) - k\rho^k(1-\rho)}{1-\rho} \right], & \text{if } \rho \neq 1 \\ \frac{k}{2}, & \text{if } \rho = 1 \end{cases} \\
&= \begin{cases} \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}, & \text{if } \rho \neq 1 \\ \frac{k}{2}, & \text{if } \rho = 1 \end{cases}
\end{aligned}$$

ii. Expected queue length i.e.

$$L_q = \sum_{n=1}^k (n-1)P_n = \sum_{n=1}^k nP_n - (1-P_0)$$

$$= L_s - (1 - P_0)$$

Note: The relationships $W_s = \frac{L_s}{\lambda}$ and $W_q = \frac{L_q}{\lambda}$ as already given for model M/M/1 are not valid here. This is because no arrivals are allowed to join the system once the maximum allowable length is reached. However, if λ is replaced by $\lambda^1 = \lambda(1 - P_k)$ which is the effective rate of arrival, then

$$W_s = \frac{L_s}{\lambda^1}, W_q = W_s - \frac{1}{\mu}$$

Remark: For M/M/1/K model, the assumption $\lambda < \mu$ is not necessary for deriving steady state results. The steady-state probabilities can be obtained for $\lambda = \mu$ also and in this case,

$$P_n = \frac{1}{k+1}, n = 1, 2, \dots, N$$

and

$$L_s = \frac{k}{2}, W_s = \frac{k+1}{2\mu}, W_q = \frac{k-1}{2\mu}, L_q = \frac{k(k-1)}{2(k+1)}$$

7.3.2. Example. At a railway station, only one train is handled at a time. The railway yard is sufficient only for two trains to wait while others are given a signal to leave the station. Trains arrive at an average rate of 6 per hour and the railway station can handle them on an average of 12 per hour. Assuming Poisson arrivals and exponential service time distributions, find the steady-state probabilities for various number of trains in the system. Also find the average waiting time of a new train coming in the system.

Solution. Here $\lambda = 6$ per hour, $\mu = 12$ per hour, $k = 2$, $\rho = \frac{\lambda}{\mu} = \frac{1}{2}$

Steady-state probabilities are:

$$P_0 = \frac{1-\rho}{1-\rho^{2+1}} = \frac{1-\rho}{1-\rho^3} = \frac{4}{7}$$

$$P_n = \rho^n \cdot \rho_0; n = 0, 1, 2, \dots$$

$$= \frac{1}{2^n} \cdot \frac{4}{7}$$

$$\text{Now, } \sum_{n=0}^2 P_n = 1 \Rightarrow P_0 \left[1 + \frac{1}{2} + \frac{1}{4} \right] = 1 \Rightarrow P_0 = \frac{4}{7}$$

$$\therefore P_n = \frac{1}{2^n} \cdot \frac{4}{7}; n = 0, 1, 2$$

The average waiting time of an incoming train

$$W_s = \frac{L_s}{\lambda^1}, \text{ where } L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}$$

$$\begin{aligned}
&= \frac{\frac{1}{2}}{1 - \frac{1}{2}} - \frac{3\left(\frac{1}{2}\right)^3}{1 - \left(\frac{1}{2}\right)^3} \\
&= 1 - \frac{\frac{3}{8}}{\frac{7}{8}} = \frac{4}{7} \\
\lambda' &= \lambda[1 - P_2] = 6\left[1 - \frac{1}{4} \cdot \frac{4}{7}\right] = \frac{36}{7} \\
\therefore W_s &= \frac{\frac{4}{7}}{\frac{36}{7}} = \frac{1}{9} \text{ hours}
\end{aligned}$$

7.4. M/M/C/K QUEUEING MODEL

In this system, the maximum size of a queue is k and the number of servers is C . For example, a car servicing station may have only facilities for offering services to C cars at a time. However, because of space limitation, the station can accept only k cars at any one point of time of servicing C ($<k$) cars.

The queuing system can be treated as a special case of the birth death process where

$$\lambda_n = \begin{cases} \lambda, & 0 \leq n < k \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \mu_n = \begin{cases} n\mu, & 0 \leq n < C \\ C\mu, & C \leq n \leq k \end{cases}$$

Hence, the steady state probabilities are given by

$$\begin{aligned}
P_n &= \begin{cases} \frac{\lambda \cdot \lambda \cdot \lambda \dots \lambda}{\mu \cdot 2\mu \dots n\mu} \cdot P_0; & 0 \leq n < C \\ \frac{\lambda \cdot \lambda \dots \lambda}{\mu \cdot 2\mu \dots C\mu \cdot \frac{(C\mu \cdot C\mu \dots C\mu)}{n-C \text{ times}}} P_0, & C \leq n \leq k \end{cases} \\
&= \begin{cases} \frac{(\lambda/\mu)^n}{n!} P_0, & C \leq n \leq k \\ \frac{(\lambda/\mu)^n}{C!C^{n-C}}, & n > k \end{cases} \quad \text{Where, } P_0 \text{ is given by } \sum_{n=0}^k P_n = 1 \\
\Rightarrow \sum_{n=0}^C \frac{(\lambda/\mu)^n}{n!} P_0 + \sum_{n=C+1}^k \frac{(\lambda/\mu)^n}{C!C^{n-C}} P_0 = 1
\end{aligned}$$

$$\Rightarrow P_0 = \left[\sum_{n=0}^C \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \sum_{n=C+1}^k \frac{C^C}{C!} \left(\frac{\lambda}{C\mu}\right)^n \right]^{-1}$$

$$= \left[\sum_{n=0}^C \frac{C^n \rho^n}{n!} + \sum_{n=C+1}^k \frac{C^C \rho^n}{C!} \right]^{-1}, \text{ where } \rho = \frac{\lambda}{C\mu}$$

7.4.1. Operating Characteristics/Performance Measures of the M/M/C/K Queueing Model

1. $L_s = \sum_{n=1}^k nP_n$
2. $L_q = \sum_{n=C+1}^k (n - C)P_n$
3. $W_s = \frac{L_s}{\lambda'}$, where $\lambda' = \lambda(1 - P_k)$
4. $W_q = \frac{L_q}{\lambda'} W$, where $\lambda' = \lambda(1 - P_k)$

7.4.2. Example. A car servicing station has two bays where service can be offered simultaneously. Due to a limitation in space, only four cars are accepted for servicing. The arrival pattern is Poisson with 12 cars per day. The service time for both the bays is exponentially distributed with $\mu = 8$ cars per day per bay. Find the average number of cars in the service station, the average number of cars waiting to be serviced and the average time a car spends in the system.

Solution. Here, $\lambda = 12, \mu = 8, C = 2$

$$\rho = \frac{\lambda}{C\mu} = \frac{12}{2 \times 8} = \frac{3}{4}, k = 4$$

Therefore, $P_0 = \left[\sum_{n=0}^C \frac{C^n \rho^n}{n!} + \sum_{n=C+1}^k \frac{C^C \cdot \rho^n}{C!} \right]^{-1}$

$$= \left[\sum_{n=0}^2 \frac{2^n \cdot \left(\frac{3}{4}\right)^n}{n!} + \sum_{n=3}^4 \frac{2^2 \left(\frac{3}{4}\right)^n}{2!} \right]^{-1}$$

$$= \left[\left(1 + \frac{3}{2} + \frac{9}{2!} \right) + 2 \left(\frac{27}{64} + \frac{81}{256} \right) \right]^{-1}$$

$$= \left[1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32} + \frac{81}{128} \right]^{-1}$$

$$= \left[\frac{128 + 192 + 144 + 108 + 81}{128} \right]^{-1} = \frac{128}{653}$$

$$P_1 = \frac{\left(\frac{3}{2}\right)^1}{1!} \cdot P_0 = \frac{3}{2} \cdot \frac{128}{653} = \frac{192}{653}$$

$$P_2 = \frac{\left(\frac{3}{2}\right)^2}{2!} \cdot P_0 = \frac{\left(\frac{3}{2}\right)^2}{2!} \cdot \frac{128}{653},$$

$$P_3 = \frac{\left(\frac{3}{2}\right)^3}{2! \cdot 2^{3-2}} P_0 = \frac{\left(\frac{3}{2}\right)^3}{2! \cdot 2} \frac{128}{653},$$

$$P_4 = \frac{\left(\frac{3}{2}\right)^4}{2! \cdot 2^{4-2}} \frac{128}{653}.$$

Now, we can easily evaluate L_s , L_q and W_s

7.5. CHECK YOUR PROGRESS

1. Explain the model M/M/C in case of first come first serve.
2. Derive steady state equations for the model M/M/1/k.
3. Write a note on M/M/C/k queueing model.

7.6. SUMMARY

In this chapter, we have discussed a variety of queueing (waiting line) models that have been developed to make better business decisions about the resources needed to provide a service. In chapter 6, we described M/M/1 queueing model, a single server model with infinite capacity, here, we explained three waiting line models namely M/M/C, M/M/1/k and M/M/C/k queueing models. For each model formulas were presented related to steady state equations and their corresponding operating characteristics. With the help of number of examples, we have been observed that the most obvious applications of queueing models are situations in which customers arrive for services. However, queuing (or waiting line) models can be successfully applied in other fields also.

8

INVENTORY MODELS

Structure

8.1. Introduction

8.2. Meaning of Inventory Control

8.3. Economic Order Quantity (EOQ)

EOQ Model with Uniform Demand

EOQ Model with Uniform Replenishment

EOQ with Different Rates of Demands in Different Cycles

EOQ Model when Shortages are Allowed

EOQ Model with Price Breaks

8.4. Check Your Progress

8.5. Summary

8.1. INTRODUCTION

In this chapter, we shall introduce the concept of inventory and discuss different kinds of inventory models. Inventory may be defined as a stock of idle resources of any kind having an economic value kept for the purpose of future affairs. The term is generally used to indicate physical resources such as raw materials, semi-finished goods used in the production process, finished products ready for delivery to consumers, human resources such as unused labour or financial resources such as operating capital etc. stock or order to meet an expected demand in the future.

In the present chapter, we discuss the models for inventory control known as economic order quantity (EOQ) models. EOQ models help in deciding as to how much quantity should be kept in stock so that one can balance the costs of holding too much stock. Here, we discuss inventory control, various factors influencing inventory and describe models for determining the EOQ: (i) when demand is uniform; (ii) when shortages are allowed;

(iii) when replenishment is uniform; (iv) when price breaks.

8.1.1. Objectives: The objective of these contents is to get familiar reader with the concept of inventory. After studying this chapter, reader should be able to describe the following concepts like:

- (i) Inventory Control
- (ii) EOQ Model with Uniform Demand
- (iii) EOQ Model with uniform replenishment
- (iv) EOQ with different rates of demands in different cycles
- (v) EOQ Model when shortages are allowed
- (vi) EOQ Model with price breaks

8.2. MEANING OF INVENTORY CONTROL

Inventories are essential for business and maintenance of inventories costs money by way of expenses on stores, personnel, equipment, insurance etc. Thus, excess inventories are undesirable. Therefore, controlling the inventories in the most profitable way is the need of every business.

8.2.1. Necessity for Maintaining Inventory

Though inventory of materials is an idle sequence (since the materials lie idle and are not to be used immediately) almost every organisation must maintain it for efficient and smooth running of its operations, if an enterprise has no inventory of materials at all. On receiving a sales order, it will have to place order for purchase of raw materials, wait for their receipt and then start production. The customer will thus have to wait for a long time for the delivery of the goods and may turn to other suppliers resulting in loss of business for the enterprise. Maintain an inventory is necessary because of the following reasons:

1. It helps in smooth and efficient running of an enterprise.
2. It provides service to the customer at a short notice timely delivery can fetch more goodwill and order.
3. In the absence of inventory, the enterprise may have to pay high prices because of piece-wise purchasing. Maintaining of inventory may earn price discount because of bulk purchasing.
4. It reduces product cost because there is an additional advantage of batching and long production runs.
5. It acts as a buffer stock when raw materials are received late and so many sale orders are likely to be rejected.

6. Process and movement inventories (also called pipe line stocks) are quite necessary in big enterprises where significant amounts of times are required to tranship items from one location to another.
7. Bulk purchases will entail less orders and therefore fewer clerical costs.
8. It helps in maintaining economy by absorbing some of the fluctuations when the demand for an item fluctuates and is seasonal.

8.2.2. Factors Influencing Inventories

The major problem of inventory control is to answer two questions:

1. How much to order?
2. When to order?

These are answered by developing a model. An inventory model is based on the consideration of the main aspects of inventory. The varieties of factors related to these are placed below:

1. Inventory related costs

Various costs associated with inventory control are often classified as follows:

(a) Purchase (or production) cost. The cost of purchasing (or producing) a unit of an item is known as purchase (or production) cost. The purchase price will become important when quantity discounts are allowed for purchases above a certain quantity.

(b) Ordering (or set up) cost. If any item is purchased, an ordering cost is incurred each time an order is placed. This cost includes the following factors: administrative (paper work, telephone calls, postage), transportation of items ordered, receiving and inspection of goods etc. If a firm produces its own inventory instead of purchasing the same from an outside source, then production set-up costs are analogous to ordering costs.

(c) Carrying (or holding) cost. Holding cost represents the cost of maintaining inventory in stock. It includes the interest on capital, rent for space used for storage, insurance of stored equipment, depreciation, taxes, etc.

(d) Shortage (or stock out) cost. The penalty cost for running out of stock (i.e. when an item cannot be supplied on the customer's demand) is known as shortage cost. This cost includes the loss of potential profit through the sale of items demanded and loss of goodwill, in terms of permanent loss of customers and its associated lost profit in future sales.

(e) Salvage cost (or selling price). When the demand for an item is affected by its quantity in stock, the decision depends upon the underlying criterion and includes the revenue from sale of the item. Salvage cost is generally combined with the storage cost and hence is neglected.

2. Demand: Demand is the number of units required per period and may be either known exactly or known in terms of probabilities. Problems in which demand is known with certainty, are called deterministic problems, whereas problems in which demand is assumed to be a random variable are called **probabilistic problems**.

3. Order cycle. The time period between placement of two successive orders is referred to as an order cycle. The order may be placed on the basis of the following two types of inventory review systems.

(a) **Continuous review.** The record of the inventory level is checked continuously until a specified point (called reorder point) is reached when a new order is placed.

(b) **Periodic review.** In this system, the inventory levels are reviewed at equal time intervals and the orders are placed at such levels.

4. Time horizon. The time period over which the inventory level will be controlled is referred to as time horizon. This can be finite or infinite depending upon the nature of demand. This is also known as the planning period over which the inventory is to be controlled. Mostly inventory planning in an enterprise is done on annual basis.

5. Lead time. The time between placing an order and its arrival in stock is known as lead time. The lead time can be either deterministic or probabilistic. If the lead time is zero, there is no need for placing an order in advance. If the lead time is known and is not equal to zero and also the demand is known, then it is required to place an order in advance-by-an amount of time equal to the lead time.

6. Re-order level. The level between maximum and minimum stock, at which the purchasing (or manufacturing) activities must start for replenishment is known as reorder level.

7. Stock replenishment: The rate at which items are added to inventory is one of the important parameters in inventory models. The actual replenishment of items may occur instantaneously or gradually. Instantaneous replacement is possible when the stock is purchased from outside sources while gradual replenishment is possible when the product is manufactured by the company.

8. Re-order quantity: This is the quantity of replenishment order. In certain cases, it is the Economic Order Quantity.

Some notations used in the models

The notations used in the development of models are as follows:

Q = number of units ordered per order

D = demand in units of inventory pr time period

N = number of orders placed per time period

TC = total inventory cost

C_0 = ordering cost (or setup cost per production run) per order

C_p = Carrying or holding cost per unit per period of time the inventory is held

C = Purchase or manufacturing price per unit inventory

C_s = shortage cost per unit of inventory

L = lead time

T = reorder cycle time i.e. time period between placement of two successive orders as a fractional part of standard time horizon

r_p = replenishment rate at which lot size Q is added to the inventory

8.3. ECONOMIC ORDER QUANTITY (EOQ)

‘Economic Order’ Quantity or ‘Economic lot size’ was first developed by Ford W Harris in 1913 in order to balance costs of holding too much stock against that of ordering in small quantities too frequently.

Economic order quantity is the size of the order representing standard quality of material and is the one for which the aggregate of the costs of procuring the inventory and costs of holding the inventory is minimum.

8.3.1. Model I: EOQ Model with Uniform Demand

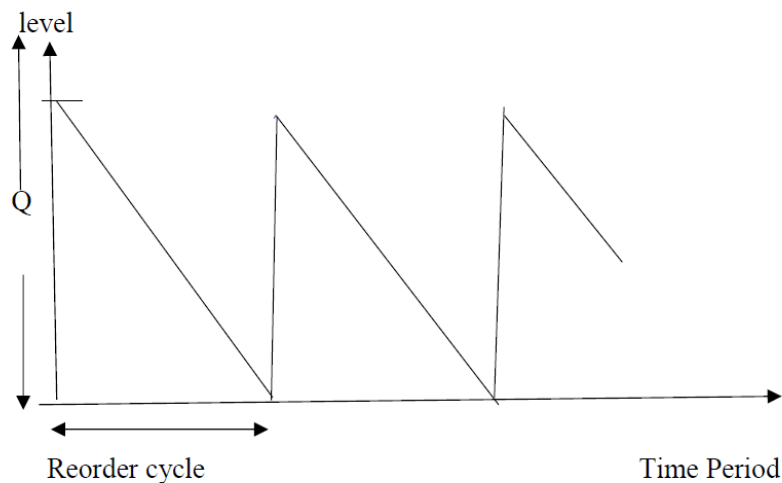
The objective of the model is to determine an optimum EOQ such that the total inventory cost is minimum. Following assumptions are made for this model:

1. Demand D is constant and known.
2. Replenishment is instantaneous i.e. the entire order quantity Q is received at one time as soon as the order is released.
3. Lead time is zero.
4. Purchase price or cost per unit is constant i.e. discounts are not allowed.

Carry cost (C_h) and ordering cost (C_o) are known and constant.

6. Shortage is not allowed.

Inventory



Since lead time is zero and replenishment is instantaneous safely stock is not required i.e. minimum level is zero. Also, demand is uniform. So, the average inventory per cycle

$$= \frac{1}{2} (\text{maximum level} + \text{minimum level}) = \frac{1}{2}(Q + O) = \frac{Q}{2}$$

Since average inventory during any cycle period is $\frac{Q}{2}$, the average inventory during the entire period is also $\frac{Q}{2}$.

So, carrying cost = average units in inventory \times carrying cost per unit $= \frac{Q}{2} C_h$

Ordering cost = number of orders \times ordering cost per order

$$= N \times C_o$$

$$= \frac{D}{Q} C_o$$

Total variable inventory cost is then given by

$$TC = \frac{D}{Q} C_o + \frac{Q}{2} C_h$$

Differentiating w.r.t. Q , we get

$$\frac{d}{dQ}(TC) = -\frac{DC_o}{Q^2} + \frac{C_h}{2}$$

$$\frac{d^2}{dQ^2}(TC) = \frac{2DC_o}{Q^3}$$

For maxima or minima,

$$\frac{d}{dQ}(TC) = 0$$

$$\text{i.e. } -\frac{DC_o}{Q^2} + \frac{C_h}{2} = 0$$

$$\text{i.e. } \frac{C_o D}{Q} = \frac{C_h Q}{2}$$

$$\text{i.e. } Q = \sqrt{\frac{2DC_o}{C_h}}$$

$$\text{At } Q = \sqrt{\frac{2DC_o}{C_h}}, \frac{d^2}{dQ^2}(TC) > 0$$

$$Q = \sqrt{\frac{2DC_o}{C_h}}$$

Hence TC is minimum when

$$Q^* = \sqrt{\frac{2DC_o}{C_h}}$$

\therefore EOQ is given as

$$(N^*) = \frac{D}{Q^*}$$

Optimum number of orders placed per time period

Minimum total variable inventory cost

$$\begin{aligned} &= \frac{D}{Q^*} C_o + \frac{Q^*}{r^2} C_h = \sqrt{2DC_o C_h} \\ &= \sqrt{2 \times \text{demand rate} \times \text{ordering cost} \times \text{holding cost}} \end{aligned}$$

Optimum length of time between orders $= \frac{Q^*}{D}$

$$= \frac{1}{D} \sqrt{\frac{2DC_o}{C_h}} = \sqrt{\frac{2C_o}{DC_h}}$$

Note: If the carrying cost is given as a percentage of average value of inventory held, then total annual carrying cost C_h may be expressed as

$$C_h = \text{cost of one unit} \times \text{inventory carrying cost in percentage} = CI$$

Hence optimum order quantity is given by

$$Q^* = \sqrt{\frac{2DC_o}{CI}}$$

8.3.1.1. Example. The demand rate of a particular item is 12000 units per year. The set-up cost per run is Rs. 350 and the holding cost is Rs. 20 per unit per month. If no shortages are allowed and the replacement is instantaneous, determine

- (i) the optimum run size,
- (ii) the optimum scheduling period, and
- (iii) minimum total expected annual cost.

Solution. Here, $D = 12000$ per year

$$C_o = \text{Rs. } 350$$

$$C_h = \text{Rs. } 0.2 \text{ per unit per month}$$

= Rs. 2.4 per unit per year

$$(i) \quad \text{Optimum lot size} = Q^* = \sqrt{\frac{2DC_o}{C_h}} = \sqrt{\frac{2 \times 12000 \times 350}{2.4}}$$

$$= \sqrt{3500000} = 1870.8 \approx 1871 \text{ units}$$

(ii) Optimum scheduling period

$$= t^* = \frac{Q^*}{D} = \frac{1871}{12000} \text{ years} = 1.87 \text{ months}$$

(iii) minimum total expected annual cost = $\sqrt{2DC_oC_h}$

$$= \sqrt{2 \times 12000 \times 350 \times 2.4}$$

$$= \sqrt{20160000} = \text{Rs. } 4490$$

8.3.1.2. Example. The annual requirement for a product is 3000 units. The ordering cost is Rs. 100 per order. The cost per unit is Rs. 10. The carrying cost per unit per year is 50% of

the unit cost. Find the EOQ. If a new EOQ is found by using ordering cost as Rs. 80, what would be the further savings in cost?

Solution. Here $D = 3000$ units per year

$$C_o = \text{Rs. } 100$$

$$C = \text{Rs. } 10, I = 30\%$$

$$C_h = CI = 10 \times \frac{30}{100} = \text{Rs. } 3 \text{ per unit per year.}$$

$$\text{Optimal lot size, } Q^* = \sqrt{\frac{2DC_o}{C_h}} = \sqrt{\frac{2 \times 3000 \times 100}{3}} = 447 \text{ units}$$

$$\text{Total inventory cost} = \sqrt{2DC_oC_h}$$

$$= \sqrt{2 \times 3000 \times 100 \times 3} = \text{Rs. } 1342 \text{ per year}$$

In the second part, we have $D = 3000$, $C_o = \text{Rs. } 100/80$, $C_h = \text{Rs. } 3$ per unit per year

$$Q^* = \sqrt{\frac{2DC_o}{C_h}} = \sqrt{\frac{2 \times 3000 \times 80}{3}} = 400 \text{ units.}$$

$$\text{Total inventory cost} = \sqrt{2DC_oC_h} = \sqrt{2 \times 3000 \times 80 \times 3} = \text{Rs. } 1200 \text{ per year}$$

Net savings = 1342 – 1200 = Rs. 142 per year.

8.3.1.3. Example. A company requires 1000 units per month. Order cost is estimated to be Rs. 50 per order. In addition to Re 1.00, the carrying costs are 10% per unit of average inventory per year. The purchase price is Rs. 10 per unit. Find the economic lot size to be ordered and the total minimum cost.

Solution. Here $D = 1000$ units per month

= 12000 units per year

$C_o = \text{Rs. } 50$ per order

$C = \text{Rs. } 10$ per unit

$C_h = 1.00 + 10\%$ of Rs. 10

= Rs. 2 per unit of average inventory

The economic lot size is given by

$$Q^* = \sqrt{\frac{2DC_o}{C_h}} = \sqrt{\frac{2 \times 12000 \times 50}{2}} = 775 \text{ units}$$

Total minimum cost = Total minimum inventory cost + Cost of material

$$= \sqrt{2DC_oC_h} + 12000 \times 10$$

$$= \sqrt{2 \times 12000 \times 50 \times 2} + 1,20,000$$

$$= 1549 + 1,20,000 = \text{Rs. } 1,21,549.$$

8.3.1.4. Example. The XYZ manufacturing company has determined from an analysis of its accounting and production data for 'part number alpha', that its cost to purchase is Rs. 36 per order and Rs. 2 per part. Its inventory carrying charge is 9 per cent of the average inventory. The demand of this part is 10,000 units per annum. Determine

(i) What should be the economic order quantity?

(ii) What is the optimum number of orders?

(iii) What is the optimum number of days supply per optimum order?

Solution. Here, demand per annum, $D = 10,000$

Ordering cost per order, $C_o = \text{Rs. } 36$

Cost of one part, $C = \text{Rs. } 2$, $I = 9\%$

Inventory carrying cost, $C_h = \text{Re } 0.09 \times 2 = \text{Rs. } 0.18$

Total inventory cost, $TC = \text{Ordering cost} + \text{Carrying cost}$

$$= \left(\frac{10000}{Q} \right) 36 + 0.09 Q$$

Differentiate w.r.t. Q ,

$$\frac{d(TC)}{dQ} = -\frac{360000}{Q^2} + 0.09$$

Equating it to zero, we have $-\frac{360000}{Q^2} + 0.09 = 0$

$$\therefore Q^2 = \frac{360000}{0.09} = 4000000$$

$$\Rightarrow Q = 2000$$

$$\frac{d^2(TC)}{dQ^2} = \frac{720000}{Q^3}$$

$$\text{At } Q = 2000, \frac{d^2(TC)}{dQ^2} = \frac{720000}{(2000)^3} > 0$$

$\therefore TC$ is minimum when $Q = 2000$ units

Thus, EOQ is $Q^* = 2000$ units .

$$(ii) \quad \text{Optimal number of orders} = \frac{\text{Demand}}{\text{EOQ}} = \frac{10000}{2000} = 5$$

$$(iii) \quad \text{Optimal number of days supply per optimum order} = \frac{365}{5} = 73 \text{ days}$$

8.3.1.5. Exercises.

1. A company uses annually 12000 units of raw material costing Rs. 1.25 per unit. Placing each order costs Rs. 15 and the carrying costs are 15% per year per unit of average inventory. Find the economic order quantity?

$$\text{Answer. } Q^* = \sqrt{\frac{2DC_o}{C_h}} = \sqrt{\frac{2 \times 12000 \times 15}{0.15 \times 1.25}} = 1385 \text{ units.}$$

2. A manufacturer uses Rs. 10,000 worth of an item during the year. He has estimated the ordering costs is Rs. 25 per order and carrying costs as 12.5% of average inventory value. Find the optimal order size, number of orders per year, time period per order and total cost.

$$\text{Answer. } Q^* = \sqrt{\frac{2DC_o}{C_h}} = \sqrt{\frac{2 \times 10000 \times 25}{0.125}} = \text{Rs. } 2000$$

$$\text{Number of orders per year} = \frac{D}{Q^*} = 5$$

$$\text{Time period per order} = \frac{365}{5} = 73 \text{ days}$$

$$\text{Total cost, } TC^* = \sqrt{2 \cdot DC_o C_h} = \sqrt{2 \times 10000 \times 25 \times 0.125} = \text{Rs. 250 (variable cost)}$$

$$\text{Total annual cost} = \text{Rs. 10000} + \text{Rs. 250} = \text{Rs. 10,250}$$

8.3.2. Model II: EOQ with Finite Rate of Replenishment or EOQ with Uniform Replenishment

For this model, it is assumed that the production run may take a significant time to complete. Let r_d be the demand rate in units per unit of time and r_p be the replenishment rate per unit time. Assume that each cycle time t of two parts t_1 and t_2 such that

- production is continuous and constant until Q units are produced to stock, then it stops;
- the production rate r_p is greater than demand rate r_d ;
- there is no replenishment (or production) during time t_2 and the inventory is decreasing at the rate r_d per unit of time.

Let Q be the number of units produced per order cycle. Then, $t_1 = \frac{Q}{r_p}$

Inventory is building up at the rate of $(r_p - r_d)$

$$\text{Maximum inventory level} = (r_p - r_d)t$$

$$\text{Minimum inventory level} = 0$$

$$\text{Average inventory} = \frac{1}{2} [(r_p r_d)t_1 + 0] = \frac{(r_p - r_d)Q}{2r_p} = \frac{Q}{2} \left(1 - \frac{r_d}{r_p}\right)$$

$$\text{Ordering cost (or set up cost)} = \frac{D}{Q} C_o$$

$$\text{Carrying cost} = \frac{Q}{2} \left(1 - \frac{r_d}{r_p}\right) C_h$$

Total inventory cost is

$$TC = \frac{D}{Q} C_o + \frac{Q}{2} \left(1 - \frac{r_d}{r_p}\right) C_h$$

This cost will be minimum if

$$\frac{D}{Q} C_o = \frac{Q}{2} \left(1 - \frac{r_d}{r_p}\right) C_h$$

$$\frac{d}{dQ}(TC) = \frac{-D}{Q^2} C_o + \frac{1}{2} \left(1 - \frac{r_d}{r_p}\right) C_h$$

$$\frac{D}{Q^2} C_o = \frac{1}{2} \left(1 - \frac{r_d}{r_p} \right) C_h$$

$$\text{i.e. } Q^2 = \frac{2DC_o}{C_h} \left(\frac{r_p}{r_p - r_d} \right)$$

$$\text{i.e. } Q^* = \sqrt{\frac{2DC_o}{C_h} \left(\frac{r_p}{r_p - r_d} \right)}$$

8.3.2.1. Characteristics of the model

1. Optimum length of each lot size production run

$$t_1^* = \frac{Q^*}{r_p} = \sqrt{\frac{2DC_o}{C_h r_p (r_p - r_d)}}$$

2. Optimum number of production runs per year

$$N^* = \frac{D}{Q^*} = \sqrt{\frac{DC_h (r_p - r_d)}{2C_o r_p}}$$

3. Optimum production cycle time, $t^* = \frac{Q^*}{D}$

4. Total minimum inventory cost

$$\begin{aligned} TC^* &= \frac{D}{Q^*} C_o + \frac{Q^*}{2} \left(1 - \frac{r_d}{r_p} \right) C_h \\ &= \frac{DC_o \sqrt{C_h (r_p - r_d)}}{\sqrt{2DC_o r_p}} + \frac{1}{2} \sqrt{\frac{2DC_o}{C_h} \cdot \frac{r_p}{r_p - r_d}} \left(1 - \frac{r_d}{r_p} \right) C_h \\ &= \sqrt{2DC_o C_h} \left(1 - \frac{r_d}{r_p} \right) \end{aligned}$$

8.3.2.2. Example. A tyre producer makes 1200 tyres per day and sells them at approximately half that rate. Accounting figures show that the production set up cost is Rs. 1000 and carrying cost per unit is Rs. 5. If annual demand is 120000 tyres, what is the optimal lot size and how many production runs should be scheduled per year?

Solution. Annual demand, $D = 120000$ tyres

$$C_h = \text{Rs. } 5$$

$$C_o = \text{Rs. } 1000$$

$$\text{Production rate, } r_p = 1200 \text{ tyres per day}$$

Demand rate, $r_d = 600$ tyres per day

$$\begin{aligned} \text{Optimal lot size, } Q^* &= \sqrt{\frac{2DC_o}{C_h} \cdot \frac{rp}{r_p - r_d}} \\ &= \sqrt{\frac{2 \times 120000 \times 1000}{5} \times \frac{1200}{1200 - 600}} \\ &= 2000\sqrt{24} = 9797.96 = 9798 \text{ tyres.} \end{aligned}$$

Optimal number of production runs per year

$$N^* = \frac{D}{Q^*} = \frac{120000}{9798} \approx 13 \text{ runs/year}$$

8.3.2.3. Example. A contractor has to supply 10000 paper cones per day to a textile unit. He finds that when he starts production run, he can produce 25000 paper cones per day. The cost of holding a paper cone in stock for one year is 2 paisa and the set up cost of production run is Rs. 18. How frequently should production run be made?

Solution. Here $r_d = 10000$ paper cones per day

$r_p = 25000$ paper cones per day

$D = 10000 \times 365 = 3,65,0000$ cones

$C_H = \text{Rs. } 0.02$ per paper cone per year

$C_o = \text{Rs. } 18$

$$\begin{aligned} \text{Now, } Q^* &= \sqrt{\frac{2DC_o}{C_h} \left(\frac{r_p}{r_p - r_d} \right)} = \sqrt{\frac{2 \times 3650000}{0.02} \times \left(\frac{25000}{25000 - 10000} \right) \times 18} \\ &= \sqrt{3650000 \times 100 \times \frac{25}{15} \times 18} = 104642 \text{ paper cones} \end{aligned}$$

Frequency of production runs is given by

$$t^* = \frac{Q^*}{r_d} = \frac{104642}{10000} = 10.46 \text{ days}$$

Thus, production run can be made after every 10.46 days

8.3.2.4. Exercises.

1. A product is produced at the rate of 50 items per day. The demand occurs at the rate of 30 items per day. Given that set up cost per order is Rs. 1000 and holding cost per unit time is Rs. 0.05. Find the economic lot size and the associated total cost per cycle assuming that no shortage is allowed.

Answer. $Q^* = 1732$ units and total inventory cost = Rs. 34.64

2. The annual demand for a product is 100000 units. The rate of production is 200000 units per year. The set up cost per production run is Rs. 500 and the variable production cost of each item is Rs. 10. The annual holding cost per unit is 20% of its value. Find the optimum production lot size and the length of the production run.

Answer. $Q^* = 10000$ units, $t_1^* = \frac{10000}{200000} = 0.05$ years

3. An item is produced at the rate of 50 items per day. The demand occurs at the rate of 25 items per day. If the set-up cost is Rs. 100 per set up and holding cost is Re 0.01 per unit of item per day, find the economic lot size for one run, assuming that shortages are not allowed. Also find the time of cycle, length of each production run, minimum total cost per day and maximum inventory level.

Answer. $Q^* = 1000$ items, Cycle time, $t_l^* = 40$ days

Length of each production run, $t_1^* = 20$ days, Maximum inventory level = 500 items

Minimum daily cost = Rs. 5

4. A company uses 100000 units of a particular item per year. Each item costs Rs. 2. The production engineering department estimates set up cost at Rs. 25 and the accounting department estimates the holding cost as 12.5% of the value of the inventory per day.

Replenishment rate is uniform 500 units per day.

Assuming 250 working days (per replenishment purpose), calculate

- (a) Optimal setup quantity
(b) Total cost on the basis of optimal policy
(c) Optimal number of set up

Answer. (a) $Q^* = 10000$ units

(b) Total minimum cost = $TC^* + 200000 = 200500$

(c) Optimum number of set ups = $\frac{D}{Q^*} = 10$

8.3.3. Model III: Economic Order Quantity with Different Rates of Demands in Different Cycles

Here the stock will vanish at different time periods with a policy of ordering same quantity for replenishment of inventory.

Here replenishment rate is infinite, replenishment is instantaneous and shortage is not allowed.

The total demand D is specified as demand during total time period T and stock level Q is fixed.

Number of production cycles, $n = \frac{D}{Q}$

Let the demand in different time periods be D_1, D_2, \dots, D_n respectively so that total demand in time T is

$$D = D_1 + D_2 + \dots + D_n$$

Where, $T = t_1 + t_2 + \dots + t_n$

Cost of ordering in time T is given by $\frac{D}{Q} C_o$

Let C_h be the holding cost per item per unit time

Then carrying cost for time T is

$$\begin{aligned} & \frac{Qt_1}{r} C_h + \frac{Qt_2}{2} C_h + \dots + \frac{Qt_n}{2} C_h \\ &= \frac{Q}{2} C_h (t_1 + t_2 + \dots + t_n) = \frac{Q}{2} C_h T \end{aligned}$$

Total inventory cost, $TC = \frac{DC_o}{Q} + \frac{Q}{2} C_h T$

Total cost is minimum, when $\frac{DC_o}{Q} = \frac{Q}{2} C_h T$

$$\text{i.e. } Q = \sqrt{\frac{2DC_o}{C_h T}} = \sqrt{\frac{2C_o \cdot D}{C_h \cdot T}}$$

$$\therefore Q^* = \sqrt{\frac{2C_o \cdot D}{C_h \cdot T}}$$

This result is similar to the result of Model I with the only different that uniform demand is replaced by average demand.

$$\text{Here, } TC^* = \sqrt{\frac{2D}{T} C_o C_h}$$

and total minimum cost = $\sqrt{\frac{2D}{T} C_o C_h} + \text{cost of material}$

Remark: If $T = 1$ year, then results of this model are exactly same as that of model I.

8.3.4. Model IV: Economic Order Quantity when Shortages are Allowed

The assumptions of this model are same as that of model I except that shortages are allowed and shortages may occur regularly. Let C_s be the shortage cost per unit of time per unit quantity.

$$\text{Ordering cost} = \frac{D}{Q} C_o$$

Total inventory over the time period, $t = \frac{1}{2} M t_1$

Average inventory at any time = $\frac{1}{2} M t_1 / t$

Inventory holding cost = $\frac{M t_1}{2t} C_h$

Total amount of shortage over time period $t = \frac{1}{2} S t_2$

Average shortage at any time $= \frac{1}{2} \frac{S t_2}{t}$

Shortage cost $= \frac{1}{2} \frac{S t_2}{t^2} C_s$

From (1),

$$-2DC_o - M^2 C_h + C_s (2Q^2 - 2QM - Q^2 + 2QM - M^2) = 0$$

$$\text{i.e., } C_s Q^2 - M^2 (C_h + C_s) - 2DC_o = 0$$

$$\text{i.e., } C_s Q^2 - \frac{Q^2 C_s^2}{(C_h + C_s)^2} (C_h + C_s) = 2DC_o \quad [\text{using (3)}]$$

$$\text{i.e. } C_s Q^2 - \frac{Q^2 C_s^2}{C_h + C_s} = 2DC_o$$

$$\text{i.e. } Q^2 \left(\frac{C_s C_h + C_s^2 - C_s^2}{C_h + C_s} \right) = 2DC_o$$

$$\text{i.e. } Q^2 = \frac{2DC_o}{C_s C_h} (C_s + C_h)$$

$$\therefore Q^* = \sqrt{\frac{2DC_o \cdot C_h + C_s}{C_h C_s}}$$

$$\text{and } M^* = \sqrt{\frac{2DC_o \cdot C_s}{C_h C_h + C_s}}$$

Total minimum cost

$$= \frac{D}{Q^*} C_o + \frac{M^{*2}}{2Q^*} C_h + \frac{(Q^* - M^*)}{2Q^*} C_s$$

$$= \sqrt{2DC_o C_h \frac{C_s}{C_h + C_s}}$$

$$\text{Total cost, } TC = \frac{D}{Q} C_o + \frac{1}{2} M \frac{t_1}{t} C_h + \frac{1}{2} S \frac{t_2}{t} C_s$$

Using the relationship for similar triangles, we have

$$\frac{t_1}{t} = \frac{M}{Q} \quad \text{and} \quad \frac{t_2}{t} = \frac{S}{Q}$$

$$t_1 = \frac{M}{Q} t \quad \text{and} \quad t_2 = \frac{S}{Q} t$$

$$\text{So } TC = \frac{D}{Q} C_o + \frac{1}{2} \frac{M^2}{Q} C_h + \frac{(Q - M)^2}{2Q} C_s \quad \text{since } S = a - M$$

Since TC is function of two variables Q and M , so in order to determine the optimal order size and optimal shortage level, we put $\frac{\partial}{\partial Q}(TC)$ and $\frac{\partial}{\partial M}(TC)$, both equal to zero so that

$$-\frac{DC_o}{Q^2} - \frac{M^2 C_h}{2Q^2} + \frac{C_s}{2} \left(\frac{Q \cdot 2(Q-M) - (Q-M)^2 \cdot 1}{Q^2} \right) = 0 \quad (1)$$

and

$$\frac{M}{Q} C_h + \frac{2C_s}{2Q} (Q-M)(-1) = 0 \quad (2)$$

From (2),

$$\begin{aligned} MC_h + C_s(-Q) + MC_s &= 0 \\ \Rightarrow M &= Q \left(\frac{C_s}{C_h + C_s} \right) \quad (3) \end{aligned}$$

8.3.4.1. Example. A contractor undertakes to supply Diesel engines to a truck manufacturer at a rate of 20 engines per day. The penalty in the contract is Rs. 100 per engine per day late for missing the scheduled delivery date. The cost of holding an engine in stock for one month is Rs. 150. His production process is such that each month (30 days) he starts producing a batch of engines through the shops and all these are available for supply after the end of the month. Determine the maximum inventory level at the beginning of each month.

Solution. Here, $C_h = \text{Rs. } \frac{150}{30}$ per engine per day = Rs. 5 per engine per day

$C_s = \text{Rs. } 100$ per engine per day

$D = 20$ engines per day

$t^* = 30$ days

\therefore Optimum inventory level at the beginning of each month

$$= M^* = \frac{C_s}{C_h + C_s}, Q^* = \frac{C_s}{C_h + C_s} \cdot Dt^* = \frac{100}{5+100} \times 20 \times 30 = 571 \text{ engines}$$

8.3.4.2. Example. A dealer supplies you the following information with regard to a product dealt in by him.

Annual demand = 10,000 units

Ordering cost = Rs. 10 per order, Price = Rs. 20 per unit

Inventory carrying cost = 20% of the value of inventory per year

The dealer is considering the possibility of allowing some back order (stock ordered) to occur. He has estimated that the annual cost of back ordering will be 25% of the value of inventory.

- (i) What is the optimal number of units of product he should buy in one lot?
- (ii) What quantity of product should be allowed to be back ordered, if any?

- (iii) What would be maximum quantity of inventory at any time of the year?
 (iv) Would you recommended to allow back-ordering? If so, what would be the annual cost saving by adopting the policy of back ordering?

Sol. Here, $D = 10,000$ units

$$C_o = \text{Rs. } 10 \text{ per order}$$

$$C_h = 20\% \text{ of Rs. } 20 = \text{Rs. } 4 \text{ per unit per year}$$

$$C_s = 25\% \text{ of Rs. } 20 = \text{Rs. } 5 \text{ per unit per year}$$

- (i) (a) When stock outs are not permitted:

$$Q^* = \sqrt{\frac{2DC_o}{C_h}} = \sqrt{\frac{2 \times 10000 \times 10}{4}} = 223.6 \text{ units}$$

- (b) When back ordering is permitted

$$Q^* = \sqrt{\frac{2DC_o \cdot (C_h + C_s)}{C_h \cdot C_s}}$$

$$= \sqrt{\frac{2 \times 10000 \times 10 \left(\frac{4+5}{5}\right)}{4}} = \sqrt{90000} = 300 \text{ units}$$

- (ii) Optimal quantity of the product to be back-ordered is given by

$$S^* = Q^* \left(\frac{C_h}{C_h + C_s} \right)$$

$$300 \left(\frac{4}{4+5} \right) = 133 \text{ units}$$

- (iii) Maximum inventory level, $M^* = Q^* - S^* = 300 - 133 = 167$ units

- (iv) Minimum total variable inventory cost in case shortages is not allowed

$$= TC(223.6) = \sqrt{2DC_o C_h}$$

$$= \sqrt{2 \times 10000 \times 10 \times 4} = \text{Rs. } 894.43$$

Minimum total variable inventory cost in case shortage is allowed

$$= TC(300)$$

$$= \sqrt{2DC_o C_h} \cdot \frac{C_s}{C_h + C_s} = \sqrt{2 \times 10,000 \times 10 \times 4 \cdot \left(\frac{5}{4+5}\right)}$$

$$= \text{Rs. } 666.67$$

Since $TC(300) < TC(223.6)$, the dealer should accept the proposal for back ordering as this will result in a saving of Rs. $(894.43 - 666.67) = \text{Rs. } 22776$ per year.

8.3.4.3. Exercise. A contractor undertakes to supply diesel engines to a truck manufacturer at a rate of 25 engines per day. He finds that the cost of holding a completed engine in stock is Rs. 16 per month and there is a clause in the contract penalizing him Rs. 10 per engine per day late for missing the scheduled delivery date production of engines is in batches and each time a new batch is started, there are setup costs of Rs. 10,000. How frequently should batches be started and what should be the initial inventory level at the time, each batch is completed?

$$\text{Solution. } Q^* = \sqrt{\frac{2DC_o}{C_h} \cdot \frac{C_h + C_s}{C_s}} = \sqrt{\frac{2 \times 25 \times 10,000}{\frac{11}{30}} \left(\frac{\frac{16}{30} + 10}{10} \right)} = 994 \text{ units (app.)}$$

$$t^* = \frac{Q^*}{D} = \frac{994}{25} = 39.76 \approx 40 \text{ days}$$

8.3.5. Problems of EOQ with Price Breaks

In the real world, it is not always true that the unit cost of an item is independent of the quantity procured. Often discounts are offered for the purchase of large quantities. These discounts take the form of price breaks.

Let us consider a manufacturer, who is encountered with a problem of determining an optimum production quantity for each production run and an optimal interval between successive runs. The following conditions are assumed to hold

- (i) Demand is known and uniform
- (ii) Shortages are not allowed
- (iii) Production for supply commodities is instantaneous

Let Q be the lot size in each production run, D , total number of units produced or supplied over the time period, C_o the cost per production run, R cost of manufacturing (or purchasing) per unit and I inventory carrying charge expressed as a % of the value of the average inventory. Then total cost is given by

$$TC = \text{Purchase cost} + \text{Holding cost} + \text{Ordering cost}$$

$$= Dk + \frac{1}{2}QkI + \frac{D}{Q}C_o$$

$$\frac{d(TC)}{dQ} = \frac{1}{2}kI - \frac{DC_o}{Q^2}$$

$$\frac{d^2}{dQ^2}(TC) = \frac{2DC_o}{Q^3}$$

For maxima or minima, $\frac{d}{dQ}(TC) = 0$

$$\text{i.e., } \frac{1}{2}kI - \frac{DC_o}{Q^2} = 0$$

$$\text{i.e., } Q = \sqrt{\frac{2DC_o}{kI}}$$

$$\text{At } Q = \sqrt{\frac{2DC_o}{kI}}, \frac{d^2(TC)}{dQ^2} > 0$$

$$\text{Hence total cost is minimum when } Q = \sqrt{\frac{2DC_o}{kI}}$$

$$\text{So total minimum cost, } TC(Q^*) = Dk + \frac{1}{2}Q^*kI + \frac{D}{Q^*}C_o$$

$$= Dk + \frac{1}{2}\sqrt{\frac{2DC_o}{kI}}kI + \frac{DC_o\sqrt{kI}}{\sqrt{2DC_o}}$$

$$= Dk + \sqrt{2DC_o kI}$$

8.3.5.1. Purchase Inventory Model with One Price Break

The purchase inventory model with only one price break may be represented as follows:

Range of quantity	Purchase cost (per unit)
$0 \leq Q < b$	P_1
$b \leq Q$	P_2

Where b is the quantity at and beyond which the quantity discount applies and $P_2 < P_1$

The procedure for obtaining EOQ may be summarized in the following steps:

Step 1. Calculate optimum order quantity Q_2^* for the lowest price (highest discount) i.e.

$$Q_2^* = \sqrt{\frac{2DC_o}{P_2I}}$$

and compare it with the quantity b

If $Q_2^* < b$, calculate optimum order quantity Q_1^* for price P_1 and compare total inventory cost for $Q = Q_1^*$ with $Q = b$ which given by

$$TC(Q_1^*) = DP_1 + \frac{D}{Q_1^*}C_o + \frac{Q_1^*}{2}IP_1$$

$$TC(b) = DP_2 + \frac{D}{b}C_o + \frac{b}{2}IP_2$$

If $TC(Q_1^*) > TC(b)$, then $Q^* = b$ otherwise $Q^* = Q_1^*$.

8.3.5.1.1. Example. Find the optimum order quantity for a product for which the price breaks are as follows:

Quantity	Unit cost (Rs.)
$0 \leq Q < 500$	10.00
$500 \leq Q$	9.25

The monthly demand for the product is 200 units, the cost of storage is 2% of the unit cost and ordering cost is Rs. 350.

Solution. We are given

$$C_o = \text{Rs. } 350$$

$$D = 200 \text{ units per month}$$

$$I = 2\% = 0.02$$

$$P_1 = \text{Rs. } 10$$

$$P_2 = \text{Rs. } 9.25$$

Highest discount available is Rs. 9.25. So we compute Q_2^o as

$$Q_2^o = \sqrt{\frac{2DC_o}{IP_2}} = \sqrt{\frac{2 \times 200 \times 350}{0.02 \times 9.25}} = 870 \text{ units}$$

Since $Q_2^o > b = 500$, the optimum purchase quantity is given by $Q^o = Q_2^o = 870$ units.

8.3.5.1.2. Example. Find the optimal order quantity for a product having the following characteristics:

Annual demand = 2400 units

Ordering cost = Rs. 100

Cost of storage = 24% of the unit cost

Price break

Quantity	Unit cost (Rs.)
$0 \leq Q < 500$	10
$500 \leq Q$	9

Solution. We have $D = 2400$ units per year

$$I = 0.24$$

$$C_o = \text{Rs. } 100$$

$$P_1 = \text{Rs. } 10$$

$$P_2 = \text{Rs. } 9$$

Highest discount available is Rs. 9, so we compute Q_2^* as

$$Q_2^* = \sqrt{\frac{2DC_o}{P_2I}} = \sqrt{\frac{2 \times 2400 \times 100}{9 \times 0.24}} = 471 \text{ units}$$

Since $Q_2^* < 500$, Q_2^* is not feasible

We calculate Q_1^* as

$$Q_1^* = \sqrt{\frac{2DC_o}{P_1I}} = \sqrt{\frac{2 \times 2400 \times 100}{10 \times 0.24}} = 447 \text{ units.}$$

Now total cost corresponding to order size 447 is

$$\begin{aligned} TC(Q_1^*) &= DP_1 + \frac{D}{Q_1^*} C_o + \frac{Q_1^*}{2} IP_1 \\ &= 2400 \times 10 + \frac{2400 \times 100}{447} + \frac{447}{2} \times 0.24 \times 10 \\ &= 2400 + 536.91 + 536.4 = \text{Rs. } 25073.31 \end{aligned}$$

Total cost at price break is

$$\begin{aligned} TC(b) &= TC(500) \\ &= DP_2 + \frac{D}{b} \times C_o + \frac{b}{2} IP_2 \\ &= 2400 \times 9 + \frac{2400}{500} \times 100 + \frac{500}{2} \times 0.24 \times 9 \\ &= \text{Rs. } 22620 \end{aligned}$$

Since $TC(b) < TC(Q_1^*)$, the optimal order quantity is the price discount quantity is 500 units.

8.3.5.2. EOQ Problems with Two Price Breaks

When there are two price breaks i.e. two quantity discounts, the situation can be represented as

Range of quantity	Purchase cost per unit
$0 \leq Q < b_1$	P_1
$b_1 \leq Q < b_2$	P_2
$b_2 \leq Q$	P_3

where b_1 and b_2 are the quantities which determine this price discount. The procedure for obtaining EOQ may be summarized in the following steps:

1. Compute the optimal order quantity for the lowest price. Let it be Q_3^* .
2. If $Q_3^* \geq b_2$, the optimum order quantity is Q_3^* .
3. If $Q_3^* < b_2$, calculate Q_2^* , the optimal order quantity for the next lowest price.
4. If $b_1 \leq Q_2^* < b_2$, then compare $TC(Q_2^*)$ and $TC(b_2)$ to determine the optimum purchase quantity.
5. If $Q_2^* < b_1$, calculate Q_1^* and compare $TC(b_1)$, $TC(b_2)$ and $TC(Q_1^*)$ to determine the optimum, purchase quantity.

8.3.5.2.1. Example. Find the optimal order quantity for a product for which the price discounts are as:

Range of quantity	Unit price (Rs.)
$0 \leq Q < 500$	10.00
$500 \leq Q < 750$	9.25
$750 \leq Q$	8.75

The monthly demand for the product is 200 units, storage cost is 2% of unit cost and ordering cost is Rs. 100.

Solution. Here $D = 200$ units

$I = 2\%$ of unit cost

$C_o = \text{Rs. } 100$ per order

$P_1 = \text{Rs. } 10$

$P_2 = \text{Rs. } 9.25$

$P_3 = \text{Rs. } 8.75$

We calculate Q_3^* as

$$Q_3^* = \sqrt{\frac{2DC_o}{IP_3}} = \sqrt{\frac{2 \times 200 \times 100}{8.75 \times 0.02}} = 475 \text{ units}$$

Since $Q_3^* < b_2 = 750$, Q_3^* is not feasible.

$$\text{Now } Q_2^* = \sqrt{\frac{2DC_o}{IP_2}} = \sqrt{\frac{2 \times 200 \times 100}{0.02 \times 9.25}} = 465 \text{ units}$$

Since $Q_2^* = b_1 = 500$, we compute Q_1^*

$$Q_1^* = \sqrt{\frac{2DC_o}{IP_1}} = \sqrt{\frac{2 \times 200 \times 100}{0.02 \times 10}} = 447 \text{ units.}$$

Now,

$$\begin{aligned} TC(Q_1^*) &= DP_1 + \frac{D}{Q_1^*} C_o + \frac{Q_1^* IP_1}{2} \\ &= 200 \times 10 + \frac{200}{447} \times 100 + \frac{447}{2} \times 0.02 \times 10 \\ &= 2000 + 44.74 + 44.70 = \text{Rs. } 2089.44 \end{aligned}$$

$$TC(b_1) = TC(500)$$

$$\begin{aligned} &= DP_2 + \frac{D}{b_1} C_o + \frac{b_1}{2} IP_2 \\ &= 200 \times 9.25 + \frac{200}{500} \times 100 + \frac{500}{2} \times 0.02 \times 9.25 \\ &= 1850 + 40 + 46.25 = \text{Rs. } 1936.25 \end{aligned}$$

$$TC(b_2) = TC(750)$$

$$\begin{aligned}
&= DP_3 + \frac{D}{b_2} C_o + \frac{b_2}{2} IP_3 \\
&= 200 \times 8.75 + \frac{200}{750} \times 100 + \frac{750}{2} \times 0.02 \times 8.15 \\
&= 1750 + 26.67 + 65.62 = \text{Rs. } 1842.29
\end{aligned}$$

The lowest total inventory cost is Rs. 1842.24

So optimal order quantity is $Q^* = b_2 = 750$

8.3.5.3. Purchase Inventory Model with n Price Breaks

When there are n price breaks, the situation may be illustrated as

Range of quantity	Purchase cost per unit
$0 \leq Q < b_1$	P_1
$b_1 \leq Q < b_2$	P_2
\vdots	\vdots
$b_{n-1} \leq Q$	P_n

where b_1, b_2, \dots, b_{n-1} are the quantities which determine the price breaks. Let $Q_1^*, Q_2^*, \dots, Q_n^*$ be EOQ corresponding to prices P_1, P_2, \dots, P_n respectively. The procedure for obtaining optimum order quantity is:

1. Compute Q_n^* . If $Q_n^* \geq b_{n-1}$, then the optimum purchase quantity is Q_n^* .
2. If $Q_n^* < b_{n-1}$, then compute Q_{n-1}^* .
If $Q_{n-1}^* \geq b_{n-2}$, then optimum order quantity is determined by comparing $TC(Q_{n-1}^*)$ with $TC(b_{n-1})$.
3. If $Q_{n-1}^* < b_{n-2}$, compute Q_{n-2}^* . If $Q_{n-2}^* \geq b_{n-3}$, then optimum order quantity is determined by comparing $TC(Q_{n-2}^*)$ with $TC(b_{n-2})$ and $TC(b_{n-1})$.
4. If $Q_{n-2}^* < b_{n-3}$, compute Q_{n-3}^* .
If $Q_{n-3}^* \geq b_{n-4}$, then optimum order quantity is determined by comparing $TC(Q_{n-3}^*)$ with $TC(b_{n-3})$, $TC(b_{n-2})$ and $TC(b_{n-1})$.
5. Continue in this way until $Q_{n-j}^* \geq b_{n-(j+1)}$; $0 \leq j \leq n-1$; and then compare $TC(Q_{n-j}^*)$ with $TC(b_{n-j})$, $TC(b_{n-j+1})$, $TC(b_{n-j+2})$, \dots , $TC(b_{n-1})$.

This procedure involves a finite number of steps.

8.3.5.3.1. Example. The annual demand for a product is 500 units. The cost of storage per unit per year is 10% of the unit cost. The ordering cost is Rs. 180 for each order. The unit cost depends upon the amount ordered. The range of amount ordered and the unit cost price are as follows:

Range of amount ordered	Price per unit
$0 \leq Q < 500$	Rs. 25
$500 \leq Q < 1500$	Rs. 24.80
$1500 \leq Q < 3000$	Rs. 24.60
$3000 \leq Q$	Rs. 24.40

Find the optimal order quantity.

Solution. Here $D = 500$ units

$C_o = \text{Rs. } 180$ per order

$I = 0.10$

$b_1 = 500, b_2 = 1500, b_3 = 3000$

$P_1 = \text{Rs. } 25, P_2 = \text{Rs. } 24.80$

$P_3 = \text{Rs. } 24.60, P_4 = \text{Rs. } 24.40$

$$\begin{aligned} \text{Step 1 } Q_4^* &= \sqrt{\frac{2DC_o}{IP_4}} = \sqrt{\frac{2 \times 500 \times 180}{0.10 \times 24.40}} = \sqrt{\frac{1000 \times 10 \times 18000}{2440}} \\ &= 1000 \sqrt{\frac{180}{2440}} = 271.6 \approx 272 \text{ units} \end{aligned}$$

Since $Q_4^* < b_3$, we compute Q_3^* .

$$\begin{aligned} \text{Step 2 } Q_3^* &= \sqrt{\frac{2DC_o}{IP_3}} = \sqrt{\frac{2 \times 500 \times 180}{0.16 \times 24.60}} \\ &= 100 \sqrt{\frac{18}{246}} = 270 \text{ units} \end{aligned}$$

Since $Q_3^* < b_2$, we compute Q_2^*

$$\text{Step 3 } Q_2^* = \sqrt{\frac{2DC_o}{IP_2}} = \sqrt{\frac{2 \times 500 \times 180}{0.10 \times 24.80}} = 269 \text{ units}$$

Since $Q_2^* < b_1$, we compute Q_1^*

$$\text{Step 4 } Q_1^* = \sqrt{\frac{2 \times 500 \times 180}{0.10 \times 25}} = \sqrt{\frac{180}{25} \times 10000} = \sqrt{72000} = 268 \text{ units.}$$

Now we compute $TC(Q_1^*), TC(b_1), TC(b_2), TC(b_3)$ and compare them to get optimal order quantity.

$$\begin{aligned} TC(Q_1^*) &= DP_1 + \frac{D}{Q_1^*} C_o + \frac{1}{2} Q_1^* IP_1 \\ &= 500 \times 25 + \frac{500}{268} \times 180 + \frac{268}{2} \times 0.10 \times 25 \\ &= 12500 + 335.82 + 335 = \text{Rs. } 13170.82 \end{aligned}$$

$$\begin{aligned}
 TC(b_1) &= DP_2 + \frac{D}{b_1} C_o + \frac{1}{2} b_1 IP_2 \\
 &= 500 \times 24.80 + \frac{500}{500} \times 180 + \frac{500}{2} \times 0.10 \times 24.80 \\
 &= 12400 + 1800 + 620 = \text{Rs. } 14820
 \end{aligned}$$

$$\begin{aligned}
 TC(b_2) &= DP_3 + \frac{D}{b_2} C_o + \frac{1}{2} b_2 IP_3 \\
 &= 500 \times 24.60 + \frac{500}{1500} \times 180 + \frac{1}{2} \times 1500 \times 0.10 \times 24.60 \\
 &= 12300 + 60 + 1845 = \text{Rs. } 14205
 \end{aligned}$$

$$\begin{aligned}
 TC(b_3) &= DP_4 + \frac{D}{b_3} C_o + \frac{1}{2} b_3 IP_4 \\
 &= 500 \times 24.40 + \frac{500}{3000} \times 180 + \frac{1}{2} \times 3000 \times 0.10 \times 24.40 \\
 &= 12200 + 30 + 3660 = \text{Rs. } 15890
 \end{aligned}$$

Since $TC(Q_1^*) < TC(b_2) < TC(b_1) < TC(b_3)$,

Optimum order quantity is Q_1^* i.e. 268 units.

8.4. CHECK YOUR PROGRESS

1. Why inventory is maintained?
2. Explain the meaning of inventory control.
3. Explain the inventory model when demand rate is uniform.
4. Derive EOQ model for an inventory problem when shortages are allowed.

8.5. SUMMARY

Inventory may be defined as stock of idle resources that are stored or reserved in order to ensure smooth and efficient running of business affairs as inventories are essential for almost all business. In this chapter, we explained the concept of inventory control and EOQ. The models discussed in this chapter are:

- (i) EOQ Model with Uniform Demand,
- (ii) EOQ Model with uniform replenishment,
- (iii) EOQ with different rates of demands in different cycles,
- (iv) EOQ Model when shortages are allowed, and
- (v) EOQ Model with price breaks

9

GAME THEORY

Structure

- 9.1. Introduction
- 9.2. Some Basic Definitions
- 9.3. Two-Person Zero-Sum Game
- 9.4. Pure Strategies: Game with Saddle Points
- 9.5. The Rule of Dominance
- 9.6. Mixed Strategies: Game without Saddle Points

Algebraic Method

Graphical Method

Linear Programming Method

- 9.7. Check Your Progress
- 9.8. Summary

9.1. INTRODUCTION

In real-life, we can see a great variety of competitive situations. Game theory provides tools for analysing situations in which parties, called players, make decisions that are interdependent. This interdependence causes each player to consider the other player's possible decisions, or strategies, in formulating strategy. A solution to a game describes the optimal decisions of the players, who may have similar, opposed, or mixed interests, and the outcomes that may result from these decisions. So, one can say that it is a type of decision theory. Game theory was originally developed by John von Neumann (called the father of game theory) and his colleague Oskar Morgenstern to solve problems in economics.

In this chapter, first we define some basic terms used in game theory then we shall discuss two-person zero-sum-games (also known as rectangular games), games with saddle point in which we study minimax and maximin criterion. Also, we shall explain rules of dominance which are used to reduce the size of the payoff matrix and discuss solution methods for game without saddle point namely algebraic method, graphical method and linear programming method.

9.1.1. Objectives. The objective of these contents is to get familiar reader with game theory. After studying this chapter, reader should be able to describe the following concepts like:

- Minimax and Maximin Principle
- Pure Strategies: Game with Saddle Points
- The Rule of Dominance
- Mixed Strategies: Game without Saddle Points

9.2. SOME BASIC DEFINITIONS

Game: A competitive situation is called a game if it has the following properties

- a) There are finite numbers of participants called players.
- b) Each player has finite number of strategies available to him.
- c) Every game result in an outcome.

Player: Each participant of a game is called a player.

Number of players: If a game involves any two payers, it is called a two-person game. However, if the number of players is more than two, the game is known as n -person game.

Payoff: A quantitative measure of satisfaction, a person gets at the end of each play, is called a payoff.

Play: A play is said to occur when each player chooses one of his activities.

Strategy: The strategy for a payer is the list of all possible actions or moves available to him. Generally, two types of strategies are employed by players in a game.

(i) Pure strategy: It is a decision rule which is always used by the player to select any one particular course of action. The objective of the payer is to maximize gains or minimize losses.

(ii) Mixed strategy: When the players use a combination of strategies and each player always keep guessing as to which course of action is to be selected by the other on a particular occasion, then this is known as mixed strategy. Thus, the mixed strategy is a selection among pure strategies with fixed probabilities.

Zero-sum game. A game in which the algebraic sum of the outcomes for all the participants equals zero for every possible combination of strategies, is called a zero-sum game.

A game which is not zero-sum is called a non-zero-sum game.

Optimal strategy. A course of action or play which puts the player in the most preferred position, irrespective of the strategy of his competitors, is called optimal strategy.

Value of the game. The expected payoff when the players follow their optimal strategy is called the value of the game.

9.3. TWO-PERSON ZERO-SUM GAME

A game with only two-persons is said to be two-person zero-sum game if the gain of one player is equal to the loss of the other so that total sum is zero.

9.3.1. Payoff Matrix: In a two-person game, the payoffs in terms of gains or losses, when players select their particular strategies can be represented in the form of a matrix, called the payoff matrix of the player. If the game is zero-sum, the gain of one player is equal to the loss of the other and vice-versa. So, one player’s payoff table would contain the same amounts I payoff table of the other payer with the sign changed. If the player *A* has strategies A_1, A_2, \dots, A_n and the player *B* has strategies B_1, B_2, \dots, B_n and if a_{ij} represent the payoffs that the player *A* gains from player *B* when player *A* chooses strategy *i*, and player *B* chooses strategy *j* then payoff matrix for player *A* is given by

$$\begin{array}{c}
 \text{Player A's strategies} \\
 \begin{matrix} A_1 \\ A_2 \\ \dots \\ A_m \end{matrix}
 \end{array}
 \begin{array}{c}
 \text{Player B's strategies} \\
 \begin{matrix} B_1 & B_1 & \dots & B_n \end{matrix} \\
 \left[\begin{array}{cccc}
 a_{11} & a_{12} & \dots & a_{1n} \\
 a_{21} & a_{22} & \dots & a_{2n} \\
 \dots & \dots & \dots & \dots \\
 a_{m1} & a_{m2} & \dots & a_{mn}
 \end{array} \right]
 \end{array}$$

9.3.2. Basic Assumptions of the Game:

Rules of the game are given as follows:

- Each player has available to him a finite number of possible courses of action. The list may not be the same for each player.
- Players act rationally and intelligently.
- The decisions of both the payers are made individually, prior to the play, with no communication between them.
- One player attempts to maximize gains and the other attempts to minimize losses.
- The players simultaneously select their respective courses of action.
- The payoff is fixed and determined in advance.
- List of strategies of each player and the amount of gain or loss on an individual’s choice of strategy is known to each player in advance.

9.4. PURE STRATEGIES: GAMES WITH SADDLE POINT

Consider the payoff matrix of a game which represents payoff of player A . Now, the objective of the study is to know how these players must select their respective strategies so that they may optimize their payoff. Such a decision-making criterion is referred to as the **minimax-maximin principle**.

For payer A , the minimum value in each row represents the least gain to him if he chooses his particular strategy. He will then select the strategy that gives the largest gain among the row minimum values. This choice of player A is called the **maximin principle** and the corresponding gain is called the maximin value of the game denoted by \underline{v} .

For player B , who is assumed to be loser, the maximum value in each column represents the maximum loss to value in each column represents the maximum loss to him if he chooses his particular strategy. He will then select the strategy that gives minimum loss among the column maximum values. This choice of player B is called the **minimax principle** and the corresponding loss is called the minimax value of the game, denoted by \bar{v} .

Saddle point. A saddle point of a payoff matrix is that position in the payoff matrix where maximum of row minima coincides with the minimum of the column maxima. The saddle point need not be unique.

Value of the game. The amount of payoff at the saddle point is called the value of the game, denoted by \underline{v} .

Fair game. A game is said to be fair if $\underline{v} = 0 = \bar{v}$.

Strictly determinable game. A game is said to be strictly determinable if $\underline{v} = v = \bar{v}$.

9.4.1. Procedure to Determine Saddle Point

- Select the minimum element in each row and enclose it in a rectangle box.
- Select the maximum element in each column and enclose it in a circle.
- Find the element which is enclosed by the rectangle as well as the circle such element is the value of the game and that position is a saddle point.

9.4.2. Example. For the game with payoff matrix:

		Player B		
		B_1	B_2	B_3
Player A	A_1	-1	2	-2
	A_2	6	4	-6

Determine the optimal strategies for players A and B .

Also determine the value of game.

Is this game (i) fair? (ii) strictly determinable?

Solution. Select the row minimum and enclose it in a rectangle. Then select the column maximum and enclose it in a circle.

	B_1	B_2	B_3
A_1	-1	2	-2
A_2	6	4	-6

Saddle point is (A_1, B_3) .

Value of game = -2

Optimal strategy for A is A_1 and for B is B_3 .

The game is strictly determinable. Since value of game is not zero, the game is not fair.

9.4.3. Example. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give optimum strategies for each player in case of strictly determinable games:

(a)	Player B	(b)	Player B
Player A	$\begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix}$	Player A	$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$

Solution. (a) Payoff matrix for player A is:

	Player B		
Player A		B_1	B_2
A_1	0	2	0
A_2	-1	4	-1
	0	4	

\underline{v} (maximin value) = 0

\bar{v} (minmax value) = 0

Since $\underline{v} = -1, \bar{v} = 1$, game is not strictly determinable.

9.4.4. Example. Solve the game if payoff matrix is given by

		B_1	B_2	B_3
A_1	1	3	1	
A_2	0	-4	-3	
A_3	1	5	-1	

Solution. Select the row minimum and enclose it in a rectangle select the column maximum and enclose it in a circle.

		Player <i>B</i>		
		B_1	B_2	B_3
Player <i>A</i>	A_1	$\boxed{1}$	3	$\boxed{1}$
	A_2	0	$\boxed{-4}$	-3
	A_3	1	$\boxed{5}$	$\boxed{-1}$

We observe that there exist two saddle points at positions (1, 1) and (1, 3). Thus, the solution of the game is given by

- (i) the optimum strategy for player *A* is A_1 .
- (ii) the optimum strategies for player *B* are B_1 and B_3 .
- (iii) the value of game is 1 for *A* and *B*.

Since $v \neq 0$, the game is not fair.

9.4.5. Example. Consider the game *G* with the following payoff matrix:

		Player <i>B</i>	
		B_1	B_2
Player <i>A</i>	A_1	2	6
	A_2	-2	λ

- (a) Show that *G* is strictly determinable, whatever λ may be
- (b) Determine the value of *G*.

Solution. First, ignoring the value of λ , we determine the maximin and minimax values of the payoff matrix, as shown below:

		Player <i>B</i>		
		B_1	B_2	Row minima
Player <i>A</i>	A_1	2	6	2
	A_2	-2	λ	-2
Column maxima		2	6	

Since maximin value = 2 = minimax value, the game *G* is strictly determinable, whatever λ may be value of game *G* is 2

9.4.6. Example. For what value of λ , the game with following payoff matrix is strictly determinable?

		B_1	B_2	B_3
A_1	λ	6	2	
A_2	-1	λ	-7	
A_3	-2	4	λ	

Solution. Ignoring the value of λ , we determine the maximin and minimax values of the payoff matrix, as shown below

		B_1	B_2	B_3	Row minimum
	A_1	-5	6	2	2 ← Maximin
Player A	A_2	-1	6	-7	-7
	A_3	-2	4	1	-2
Column maximum		-1	6	2	
		↑			Minimax

Here maximin value = 2, minimax value = -1.

The value of game lies between -1 and 2.

For strictly determinable game, since maximin value equals minimax value, we must have $-1 \leq \lambda \leq 2$.

9.4.7. Exercises. Solve the games whose payoff matrices are given below.

1. Player A $\begin{matrix} & \text{Player } B \\ & \begin{matrix} \textcircled{1} & \textcircled{0} \end{matrix} \\ \begin{matrix} \boxed{4} & \textcircled{-3} \end{matrix} & \begin{matrix} 1 \\ -3 \end{matrix} \\ \begin{matrix} 4 & 1 \end{matrix} & \end{matrix}$

Answer. $v = 1$

2. Player A $\begin{matrix} & \text{Player } B \\ & \begin{matrix} \boxed{15} & \boxed{2} & \boxed{3} \end{matrix} \\ \begin{matrix} \textcircled{6} & 5 & \textcircled{7} \end{matrix} & \begin{matrix} 2 \\ 5 \\ 0 \end{matrix} \\ \begin{matrix} -7 & 4 & 0 \end{matrix} & \\ \begin{matrix} 15 & 5 & 3 \end{matrix} & \end{matrix}$

Answer. $(A_2, B_2), v = 5$

3. Player A $\begin{matrix} & \text{Player } B \\ & \begin{matrix} B_1 & B_2 & B_3 \end{matrix} \\ \begin{matrix} \textcircled{1} & 2 & \textcircled{1} \end{matrix} & \begin{matrix} 1 \\ -4 \\ -2 \end{matrix} \\ \begin{matrix} \textcircled{0} & \boxed{-4} & -1 \end{matrix} & \\ \begin{matrix} \textcircled{1} & \textcircled{3} & \boxed{-2} \end{matrix} & \end{matrix}$

Answer. $(A_1, B_1), (A_1, B_3), v = 1$ for A .

4. $\begin{matrix} & B_1 & B_2 & B_3 \\ A_1 & \begin{bmatrix} 6 & 8 & 6 \end{bmatrix} & 1 \\ A_2 & \begin{bmatrix} 4 & 12 & 2 \end{bmatrix} & -4 \end{matrix}$

Answer. $(A_1, B_1), (A_1, B_2), v = 6$

		Player B				
		I	II	III	IV	V
5.	Player A	I	II	III	IV	V
		9	3	1	8	0
		6	5	4	6	7
		2	4	4	6	7
	5	6	2	2	1	

Answer. (II, III), $v = 4$

6 Determine which of the following two-person zero-sum games are strictly determinable and fair? Give the optimum strategies for each player in case of strictly determinable games.

	(a)	Player B		(b)	Player B
		B_1 B_2			B_1 B_2
Player A		A_1 [-5 2]	Player A		A_1 [10 6]
		A_2 [-7 -4]			A_2 [8 2]

Answer. (a) (A_1, B_1) , $v = -5$, not fair (b) (A_1, B_2) , value = 6, not fair.

9.5. PRINCIPLE OF DOMINANCE

The principle of dominance is used to reduce the size of a games payoff matrix by eliminating a course of action which is so inferior to another as never to be used. Such a course of action is said to be dominated by the other. It is applicable to both pure and mixed strategy problems. However, this rule is especially useful for the evaluation of two-person zero-sum games where a saddle point does not exist.

In general, the following rules of dominance are used to reduce the size of payoff matrix.

Rule 1. If all the elements in a column are greater than or equal to the corresponding elements in another column, then that column is dominated and can be deleted from the matrix.

Rule 2. If all the elements in a row are less than or equal to the corresponding elements in another row, then that row is dominated and can be deleted from the matrix.

Rule 3. If all the elements in a column are greater than or equal to the average of the corresponding elements of two or more other columns, then that column can be deleted.

Rule 4. If all the elements in a row are less than or equal to the average of the corresponding elements of two or more other rows, then it can be deleted.

9.5.1. Example. Reduce the following game to 2×2 game using principle of dominance.

		Player B					
		I	II	III	IV	V	VI
Player A	I	4	2	0	2	1	1
	II	4	3	1	3	2	2
	III	4	3	7	-5	1	2
	IV	4	3	4	-1	2	2
	V	4	3	3	-2	2	2

Solution. Column I, II and IV are dominated by column V, so columns I, II and VI are deleted. The reduced matrix is

		Player <i>B</i>		
		III	IV	V
Player <i>A</i>	I	0	2	1
	II	1	3	2
	III	7	-5	1
	IV	4	-1	2
	V	3	-2	2

Now row I is dominated by row 2 and row 5 is dominated by row 4. Hence deleting rows I and V, we have

		Player <i>B</i>		
		III	IV	V
Player <i>A</i>	II	1	3	2
	III	7	-5	1
	IV	4	-1	2

Now none of single row (or column) dominates another row (or column). However, column V is dominated by the average of columns III and IV. Hence deleting column V, we have

		Player <i>B</i>	
		III	IV
Player <i>A</i>	II	1	3
	III	7	-5
	IV	4	-1

Now average of row II and row III gives the row (4, -1) which dominates the row IV. Hence deleting row IV, we have

		Player <i>B</i>	
		III	IV
Player <i>A</i>	II	1	3
	III	7	-5

9.5.2. Example. Reduce the following game into 2×2 game using the rules of dominance.

		Player <i>B</i>		
		B_1	B_2	B_3
Player <i>A</i>	A_1	1	7	2
	A_2	6	2	7
	A_3	5	1	6

Solution. First, we delete the column 3 as all the elements of this column are greater than that of first column after that we delete 3rd row as all the elements of row 3 are less than the corresponding elements of row 2. Hence the reduced matrix is

		Player <i>B</i>	
		<i>B</i> ₁	<i>B</i> ₂
Player <i>A</i>	<i>A</i> ₁	1	7
	<i>A</i> ₂	6	2

9.6. MIXED STRATEGIES: GAMES WITHOUT SADDLE POINT

Pure strategies are available as optimal strategies only for those games which have a saddle point. For games which do not have a saddle point can be solved by applying the concept of mixed strategies. Her, we study algebraic, graphical and linear programming method to solve mixed strategies games.

9.6.1. Algebraic Method

Consider the two-person zero-sum game with the following payoff matrix:

		Player <i>B</i>		
		Strategy <i>B</i> ₁	Strategy <i>B</i> ₂	Probability
Player <i>A</i>	<i>A</i> ₁	<i>a</i> ₁₁	<i>a</i> ₁₂	<i>p</i>
	<i>A</i> ₂	<i>a</i> ₂₁	<i>a</i> ₂₂	1- <i>p</i>
Probability		<i>q</i>	1- <i>q</i>	

If this game is to have no saddle point, the two largest elements of the matrix must constitute one of the diagonals. We have assumed this and therefore both players use mixed strategies. Our task is to determine the probabilities with which both players choose their course of action.

In this game, let player *A* play the strategies *A*₁ and *A*₂ with respective probabilities *p* and 1 - *p* and let player *B* play his strategies *B*₁ and *B*₂ with respective probabilities *q* and 1 - *q*. The expected payoffs to player *A* when *B* plays any one of his strategies *B*₁ or *B*₂ throughout the game, are given by

<i>B</i> 's Strategy	<i>A</i> 's Strategy
<i>B</i> ₁	<i>a</i> ₁₁ <i>p</i> + <i>a</i> ₂₁ (1 - <i>p</i>)
<i>B</i> ₂	<i>a</i> ₁₂ <i>p</i> + <i>a</i> ₂₂ (1 - <i>p</i>)

Now in order that player *A* is unaffected with whatever choice of strategies *B* makes, we must have

$$\begin{aligned}
 &a_{11}p + a_{21}(1 - p) = a_{12}p + a_{22}(1 - p) \\
 \Rightarrow &(a_{11} - a_{12})p + (a_{22} - a_{21})p = a_{22} - a_{21} \\
 \Rightarrow &p = \frac{a_{22} - a_{21}}{(a_{11} - a_{12}) + (a_{22} - a_{21})}
 \end{aligned}$$

$$\text{and } 1-p = \frac{a_{11} - a_{12}}{(a_{11} - a_{12}) + (a_{22} - a_{21})}$$

similarly, by equating the expected payoffs of the player B , for whatever choice of strategies player A makes, we have

$$a_{11}q + a_{12}(1-q) = a_{21}q + a_{22}(1-q)$$

This implies

$$[(a_{11} - a_{22}) + (a_{22} - a_{12})]q = a_{22} - a_{12}$$

$$\text{So } q = \frac{a_{22} - a_{12}}{(a_{11} - a_{21}) + (a_{22} - a_{12})} \text{ and } 1-q = \frac{(a_{11} - a_{21})}{(a_{11} - a_{21}) + (a_{22} - a_{12})}$$

The value of game, v is found by substituting the value of p in one of the expressions for the expected gains of A so that

$$\begin{aligned} v &= a_{11}p + a_{21}(1-p) \\ &= \frac{a_{11}(a_{22} - a_{21})}{(a_{11} - a_{12}) + (a_{22} - a_{21})} + \frac{a_{21}(a_{11} - a_{12})}{(a_{11} - a_{12}) + (a_{22} - a_{21})} \\ &= \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} - a_{12}) + (a_{22} - a_{21})} \end{aligned}$$

Hence the solution of the game is

$$A \text{ plays } (p, 1-p) \text{ where } p = \frac{a_{22} - a_{21}}{(a_{11} - a_{12}) + (a_{22} - a_{21})}$$

$$B \text{ plays } (q, 1-q) \text{ where } q = \frac{a_{22} - a_{12}}{(a_{11} - a_{12}) + (a_{22} - a_{21})}$$

$$\text{and value of game, } v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} - a_{21}) + (a_{22} - a_{12})}.$$

9.6.1.1. Example. Solve the following game:

		Player B	
		B_1	B_2
Player A	A_1	[25	5]
	A_2	[10	15]

Solution. Here maximin value = 10 and minimax value = 15.

So, game has no saddle point.

Let the player A play his first strategy A_1 with probability p , then he would play his second strategy A_2 with probability $(1-p)$. Then expected gain of A if B selects B_1 , is equal to $25p + 10(1-p)$ i.e. $10 + 15p$ and the expected gain of A if B selects strategy B_2 , is equal to $5p + 15(1-p)$ i.e. $15 - 10p$.

Now in order that the player A may be unaffected with whatever choice B makes, the optimal plan for the player A should be such that the expected payoffs for each of B 's strategies should be equal is

$$10 + 15p = 15 - 10p$$

$$\therefore p = \frac{5}{25} = \frac{1}{5} \text{ and } 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$$

Hence, the player A would play his first strategy A_1 with probability $\frac{1}{5}$ and second strategy A_2 with probability $\frac{4}{5}$.

Similarly, if the player B selects strategies B_1 and B_2 with probabilities q and $1 - q$ respectively, then the expected loss to B when A adopts the strategy A_1 , is $25q + 5(1 - q)$ and the expected loss to B when the player A adopts the strategy A_2 , is $10q + 15(1 - q)$. By equating the expected losses of player B , for whatever choice of strategies player A makes, we have

$$25q + 5(1 - q) = 10q + 15(1 - q)$$

$$\Rightarrow 20q + 5 = 15 - 5q$$

$$\Rightarrow q = \frac{10}{25} = \frac{2}{5} \text{ and } 1 - q = \frac{3}{5}$$

Hence the player B would play his strategies B_1 and B_2 with probabilities $\frac{2}{5}$ and $\frac{3}{5}$ respectively.

Value of the game = expected payoff to player A

$$= 25p + 10(1 - p)$$

$$= 25 \times \frac{1}{5} + 10 \times \frac{4}{5} = 13.$$

9.6.1.2. Example. For the following game:

		Firm B			
		B_1	B_2	B_3	B_4
Firm A	A_1	35	65	25	5
	A_2	30	20	15	0
	A_3	40	50	0	10
	A_4	55	60	10	15

Determine the optimal strategies for each firm and value of the game.

Solution. Since maximin value = 10 and minimax value = 15, there is no saddle point.

We apply rules of dominance to reduce the size of payoff matrix. Since each element of second row is less than the corresponding elements of first row, second row is dominated by first row. So, deleting the second row, the reduced matrix becomes

		Firm B			
		B_1	B_2	B_3	B_4
Firm A	A_1	35	65	25	5
	A_3	40	50	0	10
	A_4	55	60	10	15

In the reduced matrix, each element of second column is more than the corresponding elements in first column, so second column is dominated by first column. Thus after deleting the second column, the reduced matrix becomes

		Firm B		
		B_1	B_3	B_4
Firm A	A_1	35	25	5
	A_3	40	0	10
	A_4	55	10	15

Further second row is dominated by third row, so we delete second row to get reduced matrix as

		Firm B		
		B_1	B_3	B_4
Firm A	A_1	35	25	5
	A_4	55	10	15

Now column one is dominated by column two. So, we delete column one and reduced matrix becomes

		Firm B	
		B_3	B_4
Firm A	A_1	25	5
	A_4	10	15

Now we solve the game by algebraic method in the same manner as we did in example 9.6.1.1. Thus, firm A would select strategy A_1 with probability $\frac{1}{5}$ and strategy A_4 with probability $\frac{4}{5}$ and firm B would select strategy B_3 with probability $\frac{2}{5}$ and strategy B_4 with probability $\frac{3}{5}$ and Value of game =

$$25p + 10(1 - p) = 25 \cdot \frac{1}{5} + 10 \cdot \frac{4}{5} = 5 + 8 = 13.$$

9.6.1.3. Example. In a game of matching coins with two players, suppose one player wins Rs. 2 when there are two heads and wins nothing when there are two tails, and losses Re. 1 when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game.

Solution. Let the two players be A and B. Then the payoff matrix for player A is

		Player <i>B</i>	
		H	T
Player <i>A</i>	H	2	-1
	T	-1	0

Here maximin value (\underline{v}) = -1; minimax value (\bar{v}) = 2.

Since $\underline{v} \neq \bar{v}$, given game has no saddle point. Let the player *A* plays H with probability p and T with probability $1 - p$. Then *A*'s expected gains when *B* plays H and T respectively, are $2p + (-1)(1 - p)$ and $-p + 0(1 - p)$.

For best strategy of *A*, we have

$$2p + (-1)(1 - p) = -p.$$

so that $p = \frac{1}{4}$ and $1 - p = \frac{3}{4}$. Therefore, best strategy for player *A* is to play H and T with probabilities $\frac{1}{4}$ and $\frac{3}{4}$ respectively.

For player *B*, let the probability of the choice of H be q and that of T be $1 - q$. For best strategy of *B*, we have

$$2q + (-1)(1 - q) = (-1)q + 0(1 - q) \text{ so that } q = \frac{1}{4} \text{ and } 1 - q = \frac{3}{4}.$$

Hence player *B* should play H and T with probabilities $\frac{1}{4}$ and $\frac{3}{4}$ respectively.

Value of game = $2p + (-1)(1 - p) = -\frac{1}{4}$ for player *A*.

9.6.1.4 Exercises. Solve the following games without saddle points.

$$1. \quad \begin{array}{cc} & B_1 & B_2 \\ A_1 & \left[\begin{array}{cc} 2 & 5 \end{array} \right] \\ A_2 & \left[\begin{array}{cc} 7 & 3 \end{array} \right] \end{array}$$

Answer. $\left(\frac{4}{7}, \frac{3}{7}\right)$ for player *A*, $\left(\frac{2}{7}, \frac{5}{7}\right)$ for player *B* and Value of game = 0.

2. In a game of matching coins with two players, suppose *A* wins one unit of value, when there are two heads, wins nothing when there are two tails and loses $\frac{1}{2}$ unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of game to *A*.

$$\begin{array}{cc}
 & \begin{array}{cc} \text{H} & \text{T} \end{array} \\
 \begin{array}{c} \text{H} \\ \text{T} \end{array} & \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}
 \end{array}$$

Answer. Payoff matrix for A is

Optimal strategies for two players are $\left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{4}, \frac{3}{4}\right)$ and value of the game is $-\frac{1}{8}$.

3. The firms are competing for businesses under the conditions so that one firm’s gain is another firm’s loss. Firm A’s payoff matrix is given below:

		Firm B		
		No advertising	Medium advertising	Heavy advertising
Firm A	No Adv.	10	5	-2
	Medium Adv.	13	12	15
	Heavy Adv.	16	14	10

Suggest the optimum strategies for the firms.

Answer. $\left(0, \frac{4}{7}, \frac{3}{7}\right), \left(0, \frac{5}{7}, \frac{2}{7}\right), v = \frac{90}{7}$.

9.6.2. Graphical method

The graphical method is useful for solving two person–zero–sum–game. A Game having saddle point can be easily solved, so, we consider games without saddle point, where the payoff matrix is of size $2 \times n$ or $m \times 2$.

Optimal strategies for both the players assign no–zero probabilities to the same number of pure games. Therefore, if one player has only two strategies, the other will also use the same number of strategies. Hence, this method is useful in finding out which of the two strategies can be use. Consider the following $2 \times n$ payoff matrix of a game without saddle point.

Player A	B_1	B_2	B_3	Probability
A1	a_{11}	a_{12}	a_{13}	p_1
A2	a_{21}	a_{22}	a_{23}	p_2
Probability	q_1	q_2	q_3	

To solve this game, we draw two vertical lines at unit distance, for representing $p_1=0$ and $p_2=0$ where $p=(p_1, p_2)$ is the strategy of A and $q=(q_1, q_2, \dots, q_n)$ is the strategy of B.

We now draw n line segments joining the points $(0, a_{2j})$ and $(1, a_{1j})$, $j= 1, 2, \dots, n$ but excluding the end points. The lower envelope of these lines gives the minimum expected gain of A as a function of p_1 . The highest point o of this lower boundary of these lines will give maximum of the minimum gain of A, i.e. maximin of A.

Now, the two strategies of player B corresponding to those lines which pass through the maximum point can be determined. It helps in reducing the size of the game to (2×2) , which can be easily solved by any of the methods discussed earlier.

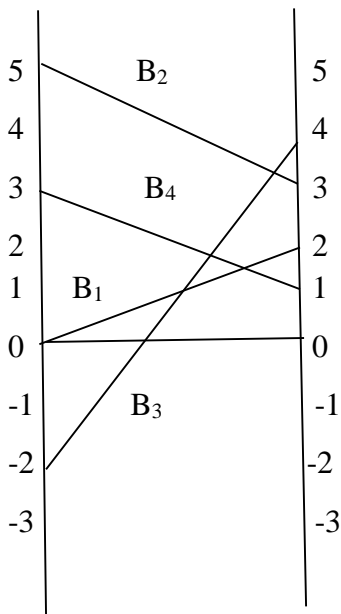
Remark: The $(m \times 2)$ games are also treated in the same way except the upper boundary of the straight lines corresponding to B's expected payoff will give the maximum expected payoff to player B and the lower point on this boundary will then give the minimum expected payoff (minimax value) and the optimum value of probability q_1 and q_2 .

9.6.2.1. Example. Use graphical method in solving the following game and find the value of the game

Solve the following game:

		Player B			
		B ₁	B ₂	B ₃	B ₄
Player A	A ₁	0	5	-2	3
	A ₂	2	3	4	1

Solution. Since $\text{maximin } a_{ij} = 3 < \text{minimax } a_{ij} = 4$, the game is to be solved by mixed strategies. We therefore use the graphical method to reduce this to a 2×2 by game as follows:



We join the points 0, 5, -2 and 3 on the left line given by $p_1=0$ to the points 2, 3, 4 and 1 on the right line given by $p_2=0$ respectively. Clearly the highest point of the lower envelop determines the strategies B_1 and B_4 .

So, the reduced game is:

		Player B	
		B_1	B_4
Player A	A_1	0	3
	A_2	2	1

Solving this game, we get

$$p_1^* = 1/4, p_2^* = 3/4; q_1^* = 1/2, q_4^* = 1/2$$

Hence the required solution is

Note: An $m \times n$ game is solvable if it has a saddle point but if it has no saddle point, it cannot be solved by graphical method unless it is reducible to the form $m \times 2$ or $2 \times n$ game by the dominance principle.

9.6.3. Linear Programming Method

The two person – zero – sum - game can also be solved by linear programming. The major advantages of using the programming technique is to solve mixed-strategy games of larger dimension payoff matrix.

To illustrate the transformation of the game problem to a linear programming problem, consider a payoff matrix of size $m \times n$. Let a_{ij} be the element in the i th row and j th column of game payoff matrix, and letting the probabilities of m strategies ($i = 1, 2, 3, \dots, m$) for player A, for each of player B's strategies will be

$$V = \sum_{i=1}^m p_i \cdot a_{ij}, \quad j = 1, 2, \dots, n$$

The aim of player A is to select a set of strategies with probability p_1 , the value of the game to the played A for all strategies by the player B must be at least equal to V . Thus, to miximize the minimum expected gains, it is necessary that

$$\begin{aligned} a_{11}p_1 + a_{12}p_2 + \dots + a_{m1}p_m &\geq V \\ a_{21}p_1 + a_{22}p_2 + \dots + a_{m2} &\geq V \\ \cdot & \\ \cdot & \\ \cdot & \\ a_{1n}p_1 + a_{2n}p_2 + \dots + a_{mn}p_m &\geq V \\ p_1 + p_2 + \dots + p_m &= 1; p_i \geq 0 \text{ for all } i \end{aligned}$$

Dividing both sides of the m inequalities of the and equation by V the division is valid as long as $V > 0$. In case $V < 0$, The direction of inequality constraints must be reversed. But if $V = 0$, division would be meaningless. In this case a constraint can be added to all entries of the matrix ensuring that the value of

the game (V) for the revised matrix becomes more than zero. After optimal solution is obtained for the value of the game is obtained by subtracting the same constant value. Let $p_i/V = x_i, (\geq 0)$. Then we get,

$$\begin{aligned} a_{11} \frac{p_1}{V} + p_{21} \frac{p_2}{V} + \dots + a_{m1} \frac{p_m}{V} &\geq 1 \\ a_{12} \frac{p_1}{V} + p_{22} \frac{p_2}{V} + \dots + a_{m2} \frac{p_m}{V} &\geq 1 \\ \cdot & \\ \cdot & \\ \cdot & \\ a_{1n} \frac{p_1}{V} + p_{2n} \frac{p_2}{V} + \dots + a_{mn} \frac{p_m}{V} &\geq 1 \\ \frac{p_1}{V} + \frac{p_2}{V} + \dots + \frac{p_m}{V} &= 1 \end{aligned}$$

Since, the objective of player A is to maximize the value of the game, V which is equivalent to minimize $\frac{1}{V}$, the resulting linear programming problem can be stated as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{m1}x_m &\geq V \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{m2}x_m &\geq V \\ \cdot & \\ \cdot & \\ \cdot & \\ a_{1n}x_1 + a_{2n}x_2 + \dots + a_{mn}x_m &\geq V \end{aligned}$$

$$\text{And } x_1, x_2, \dots, x_m \geq 0, x_i = \frac{p_i}{V}; i = 1, 2, \dots, m$$

Similarly, Player B has a similar problem with the inequalities of the constraints reversed, i.e., minimize the expected loss. Since minimizing of V is equivalent to maximizing $\frac{1}{V}$, therefore, the resulting programming problem can be stated as:

$$\text{Maximize } Z_q \left(= \frac{1}{V} \right) = y_1 + y_2 + \dots + y_n$$

Subjected to the constraints

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n &\leq 1 \\ a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n &\leq 1 \\ \cdot & \\ \cdot & \\ \cdot & \\ a_{m1}y_1 + a_{m2}y_2 + \dots + a_{mn}y_n &\leq 1 \end{aligned}$$

$$\text{And } y_1, y_2, \dots, y_n \geq 0$$

$$\text{Where, } y_j = \frac{q_j}{V} \geq 0; i = 1, 2, \dots, n$$

It may be noted that the LP problem for the player B is the dual of LP problem for player A and vice-versa. Therefore, solution of the dual problem can be obtained from the primal simplex table. Since both the players $Z_p = Z_q$, the expected gain to player A in the game will be exactly equal to expected payoff to player B.

Remark: Linear programming technique require all variables to be non-negative and therefore to derive a non – negative value V of the game, the data to the problem, i.e., a_{ij} in the payoff table should be non – negative. If there are some negative elements in the payoff table, a constant to every elements of the payoff table must be added so as to make the smallest element zero; the solution to this new game give an optimal mixed strategy for the new game. The value of the original game then equals to the value of the new game minus the constant.

9.6.3.1. Example. For the following payoff matrix, transform the zero-sum game into an equivalent linear programming problem and solve it by using simplex method.

		Player B		
	Player A	B ₁	B ₂	B ₃
	A ₁	1	-1	3
	A ₂	3	5	-3
	A ₃	6	2	-2

Solution: The first step is to find out the saddle point (if any) in the payoff matrix as shown below

		Player B			
	Player A	B ₁	B ₂	B ₃	Row minimum
	A ₁	1	-1	3	-1
	A ₂	3	5	-3	-3
	A ₃	6	2	-2	-2
	Column maximum	6	5	3	← Minimax

The given game payoff matrix does not have a saddle point. Since, the maximin value is -1, therefore, it is possible that the value of game (V) may be negative or zero because $-1 < V < 1$. Thus, a constant which is at least equal to the negative of maximin value, i.e., more than -1 is added to all elements of the payoff matrix. Thus, adding a constant number 4 to all the elements of the payoff matrix, the payoff matrix becomes:

		Player B			
	Player A	B ₁	B ₂	B ₃	Probability
	A ₁	5	3	7	p_1
	A ₂	7	9	1	p_2
	A ₃	10	6	2	p_3
	Probability	q_1	q_2	q_3	

Let p_i ($i = 1,2,3$) and q_j ($j = 1,2,3$) be the probabilities of selecting strategies

A_i ($i = 1,2,3$) and B_j ($j = 1,2,3$) by players A and B, respectively.

The expected gain for player A will be as follows:

$$5p_1 + 7p_2 + 10p_3 \geq V \quad (\text{if B uses strategy } B_1)$$

$$3p_1 + 9p_2 + 6p_3 \geq V \quad (\text{if B uses strategy } B_2)$$

$$7p_1 + p_2 + 2p_3 \geq V \quad (\text{if B uses strategy } B_3)$$

$$p_1 + p_2 + p_3 = 1$$

and $p_1, p_2, p_3 \geq 0$

Dividing each inequality and equality by V, we get,

$$5\frac{p_1}{V} + 7\frac{p_2}{V} + 10\frac{p_3}{V} \geq 1$$

$$3\frac{p_1}{V} + 9\frac{p_2}{V} + 6\frac{p_3}{V} \geq 1$$

$$7\frac{p_1}{V} + \frac{p_2}{V} + 2\frac{p_3}{V} \geq 1$$

$$\frac{p_1}{V} + \frac{p_2}{V} + \frac{p_3}{V} = \frac{1}{V}$$

In order to simplify, we define new variables:

$$x_1 = p_1/V, \quad x_2 = p_2/V \quad \text{and} \quad x_3 = p_3/V$$

The problem for player A, therefore becomes,

Minimize $Z_p (=1/V) = x_1 + x_2 + x_3$ subject to the constraints

$$5x_1 + 7x_2 + 10x_3 \geq 1$$

$$3x_1 + 9x_2 + 6x_3 \geq 1$$

$$7x_1 + x_2 + 2x_3 \geq 1$$

and $x_1, x_2, x_3 \geq 0$

player B's objective is to minimize his expected losses which can be reduced to minimizing the value of the game V. Hence, the problem of player B can be expressed as follows:

Minimize $Z_q (=1/V) = y_1 + y_2 + y_3$

subject to the constraints

$$5y_1 + 7y_2 + 10y_3 \leq 1$$

$$3y_1 + 9y_2 + 6y_3 \leq 1$$

$$7y_1 + y_2 + 2y_3 \leq 1$$

and $y_1, y_2, y_3 \geq 0$

where $y_1 = q_1/V, \quad y_2 = q_2/V \quad \text{and} \quad y_3 = q_3/V.$

It may be noted that problem of player A is the dual of the problem of player B. Therefore, solution of the dual problem can be obtained from the optimal simplex table of primal.

To solve the problem of player B, introduce slack variables to convert the three inequalities to equalities. The problem becomes

$$\text{Minimize } Z_q (=1/V) = y_1 + y_2 + y_3 + 0s_1 + 0s_2 + 0s_3$$

subject to the constraints

$$5y_1 + 7y_2 + 10y_3 + s_1 = 1$$

$$3y_1 + 9y_2 + 6y_3 + s_2 = 1$$

$$7y_1 + y_2 + 2y_3 + s_3 = 1 \quad \text{and } y_1, y_2, y_3, s_1, s_2, s_3 \geq 0$$

The initial solution is shown in Table 12.7.

Table 12.7 Initial Solution

$c_j \rightarrow$		1	1	1	0	0	0		
Unit Cost c_B B $y_B(=b)$	Variables in Basis	Solution Values	$y_1 y_2 y_3 s_1 s_2 s_3$						Min. Ratio y_B/y_1
0	s_1	1	5	3	7	1	0	0	1/5
0	s_2	1	7	9	1	0	1	0	1/7
0	s_3	1	10	6	2	0	0	1	1/10 →
Z=0		z_j	0	0	0	0	0	0	
		$c_j - z_j$	1	1	1	0	0	0	

Proceeding with usual simplex method, the optimal solution is shown in Table 12.8.

Table 12.8 Optimal Solution

$c_j \rightarrow$		1	1	1	0	0	0	
Unit Cost c_B B	Variables in Basis B	Solution Values $y_B(=b)$	$y_1 y_2 y_3 s_1 s_2 s_3$					
1	y_3	1/10	2/5	0	1	3/20	-1/10	0
1	y_2	1/10	11/15	1	0	-1/60	7/60	0
0	s_3	1/5	24/5	0	0	-1/5	-3/5	1
Z=1/5		z_j	17/15	0	0	2/15	1/15	0
		$c_j - z_j$	-2/15	1	1	-2/15	-1/15	0

The optimal solution (mixed strategies) for B is: $y_1 = 0$; $y_2 = 1/10$ and $y_3 = 1/10$ and expected value of the game is: $Z = 1/V - \text{constraint } (= 4) = 5-4 = 1$.

These solution values are now converted back into the original variables; if $1/V = 1/5$ then $V=5$

$$y_1 = q_1/V, \text{ then } q_1 = y_1 \times V = 0$$

$$y_2 = q_2/V, \text{ then } q_2 = y_2 \times V = 1/10 \times 5 = 1/2$$

$$y_3 = q_3/V, \text{ then } q_3 = y_3 \times V = 1/10 \times 5 = 1/2$$

The optimal strategies for player A are obtained from the $c_j - z_j$ row of the Table 12.8.

$$x_1 = 2/15, \quad x_2 = 1/15 \text{ and } x_3 = 0$$

$$\text{Then } p_1 = x_1 \times V = (2/15) \times 5 = 2/3; \quad p_2 = x_2 \times V = (1/15) \times 5 = 1/3$$

$$p_3 = x_3 \times V = 0$$

Hence, the probabilities of using strategies by both the players are:

Player A: $(2/3, 1/3, 0)$, Player B: $(0, 1/2, 1/2)$ and Value of the game is $V = 1$.

9.6.3.2. Exercises.

1. A soft drink company calculated the market share of two products against its major competitor having products and found out the impact of additional advertisement in any one of its products against the other

Company A	Company B		
	B ₁	B ₂	B ₃
A ₁	6	7	15
A ₂	20	12	10

What is the best strategy for the company as well as competitor? What is the payoff obtained by the company and the competitor in the long run? Use graphical method to obtain the solution.

Answer. Company A: $(2/3, 1/3, 0)$, Company B: $(7/12, 5/12)$ and $V = 1/3$.

2. In a town there are only two discount stores ABC and XYZ. Both stores run annual pre – Diwali sales. Sales are advertised through local newspapers with the aid of an advertising firm. ABC stores constructed following payoff in units of Rs 1,00,000. Find the optimal strategies for both stores and the value of the game.

Store ABC	Store XYZ		
	B ₁	B ₂	B ₃
A ₁	1	-2	1
A ₂	-1	3	2
A ₃	-1	-2	3

Answer. Add 2 (absolute value of the smallest negative payoff value) to each element of the payoff matrix. Then formulate an LP model for store XYZ. Optimal values of decision variables are: $y_1 = y_2 = y_3 = 1/6$ and $Z = 1/2 = 1/V$ or $V = 2$. Subtract 2 from $V=2$ to get $V=0$.

9.7. CHECK YOUR PROGRESS

1. What do you mean by a game in game theory? What are the assumptions made in game theory?
2. Explain maximin and minimax criterion used in game theory.
3. Define the following terms:
 - i. Saddle point,
 - ii. Two-person zero -sum game,
 - iii. Strictly determinable game,
 - iv. Value of the game.
4. Explain the rules of dominance.
5. Explain the algebraic method for solving rectangular games.

9.8. SUMMARY

In this chapter, we discussed about basic concepts/terminologies of game theory such as payoff matrix, pure strategy, mixed strategy etc. We explain basic assumptions of the game and discuss two-person zero-sum game i.e. rectangular games. In certain cases, we observed that there is no pure strategy solution for a game i.e. no saddle point exists. In all such cases, one can use methods which involve concept of mixed strategy. Here, we study three methods namely algebraic, graphical and linear programming methods for solving the problems having no saddle point. However, game theory is limited in scope as it has been capable of analysing simple competitive situations.